

# Applied Econometrics

## Lecture 10: Binary Choice Models

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## 1. Introduction

The methods discussed thus far in the course are well suited for modelling a **continuous, quantitative** variable - e.g. economic growth, the log of value-added or output, the log of earnings etc.

Many economic phenomena of interest, however, concern variables that are not continuous or perhaps not even quantitative.

- What characteristics (e.g. parental) affect the likelihood that an individual obtains a higher degree?
- What determines labour force participation (employed vs not employed)?
- What factors drive the incidence of civil war?

Today we will discuss **binary choice models**. These are central models in applied econometrics. Obviously binary choice models are useful when our outcome variable of interest is binary - a common situation in applied work. Moreover, the binary choice model is often used as an **ingredient** in other models. For example:

- In propensity score matching models (to be covered in lectures 11-12), we identify the average treatment effect by comparing outcomes of treated and non-treated individuals who, a priori, have similar probabilities of being treated. The probability of being treated is typically modelled using probit.
- In Heckman's selection model, we use probit in the first stage to predict the likelihood that someone is included (selected) in the sample. We then control for the likelihood of being selected when estimating our equation of interest (e.g. a wage equation)

The binary choice model is also a good starting point if we want to study more complicated models. Later on in the course we will thus cover **extensions** of the binary choice model, such as models for multinomial or ordered response, and models combining continuous and discrete outcomes (e.g. corner response models). These extensions will be discussed in lectures 13-14. Finally, in lecture 15 we will see

how these models can be modified to take into account unobserved heterogeneity, when **panel data** are available.

**Useful references for this lecture:**

Wooldridge, J. (2002) *Econometric Analysis of Cross Section and Panel Data*. Chapters 15.1-7 (read carefully).

Angrist, Joshua and Jörn-Stefen Pischke (2009). *Mostly Harmless Econometrics. An Empiricist's Companion*. Chapter 3.4.2 (skim).

Kingdon, G. (1996) 'The quality and efficiency of private and public education: a case-study of urban India,' *Oxford Bulletin of Economics and Statistics* 58: 57-81 (most of the empirical examples below will draw on this paper).

In addition, I will draw on material presented in the following three papers:

Martins, M. F. O. 2001. "Parametric and semiparametric estimation of sample selection models: an empirical application to the female labour force in Portugal," *Journal of Applied Econometrics* 16, pp. 23-39.

Pagan, Adrian. 2002. "Learning about Models and their Fit to Data," *International Economic Journal* 16:2, pp.1-18.

Pagan, Adrian and Frank Vella. 1989. "Diagnostic Tests for Models Based on Individual Data: A Survey," *Journal of Applied Econometrics* 4: S29-S59.

These papers are not required reading.

## 2. Binary Response Models

Whenever the variable that we want to model is binary, it is natural to think in terms of **probabilities**, e.g.

- 'What is the probability that an individual with such and such characteristics owns a car?'
- 'If some variable  $X$  changes by one unit, what is the effect on the probability of owning a car?'

When the dependent variable  $y$  is binary, it is typically equal to one for all observations in the data for which the event of interest has happened ('success') and zero for the remaining observations ('failure').

Provided we have a random sample, the sample mean of this binary variable is an unbiased estimate of the unconditional probability that the event happens. That is, letting  $y$  denote our binary dependent variable, we have

$$\Pr(y = 1) = E(y) = \frac{\sum_i y_i}{N},$$

where  $N$  is the number of observations in the sample.

Estimating the unconditional probability is trivial, but usually not the most interesting thing we can do with the data. Suppose we want to analyse what factors 'determine' changes in the probability that  $y$  equals one. Can we use the classical linear regression framework to this end?

## 3. Estimation by OLS: The Linear Probability Model

Consider the linear regression model

$$\begin{aligned} y &= \beta_1 + \beta_2 x_2 + \dots + \beta_K x_K + u \\ &= \mathbf{x}\boldsymbol{\beta} + u, \end{aligned} \tag{3.1}$$

where  $\boldsymbol{\beta}$  is a  $K \times 1$  vector of parameters,  $\mathbf{x}$  is a  $N \times K$  matrix of explanatory variables, and  $u$  is a residual. For now, we will assume that the residual is uncorrelated with the regressors, i.e. that endogeneity is not a problem. This allows us to use OLS to estimate the parameters of interest.

- To interpret the results, note that if we take expectations on both sides of the equation above we obtain

$$E(y|\mathbf{x}; \boldsymbol{\beta}) = \mathbf{x}\boldsymbol{\beta}.$$

- Now, just like the unconditional probability that  $y$  equals one is equal to the unconditional expected value of  $y$ , i.e.  $E(y) = \Pr(y = 1)$ , the conditional probability that  $y$  equals one is equal to the conditional expected value of  $y$ :

$$\begin{aligned} \Pr(y = 1|\mathbf{x}) &= E(y|\mathbf{x}; \boldsymbol{\beta}), \\ \Pr(y = 1|\mathbf{x}) &= \mathbf{x}\boldsymbol{\beta}. \end{aligned} \tag{3.2}$$

Because probabilities must sum to one, it must also be that

$$\Pr(y = 0|x) = 1 - \mathbf{x}\boldsymbol{\beta}.$$

- Equation (3.2) is a **binary response model**. In this particular model the probability of success (i.e.  $y = 1$ ) is a **linear** function of the explanatory variables in the vector  $\mathbf{x}$ . This is why using OLS with a binary dependent variable is called the **linear probability model** (LPM).

Notice that in the LPM the parameter  $\beta_j$  measures the change in the probability of 'success', resulting from a change in the variable  $x_j$ , holding other factors fixed:

$$\Delta \Pr(y = 1|\mathbf{x}) = \beta_j \Delta x_j.$$

This can be interpreted as a partial effect on the probability of 'success'.

EXAMPLE: Modelling the probability of going to a private, unaided school (PUA) in India.<sup>1</sup> See appendix, Table 1a.

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<sup>1</sup>The data for this example are taken from the study by Kingdon (1996).

- Summary statistics
- LPM. Interpret.

### 3.1. Shortcomings of the Linear Probability Model

Clearly the LPM is straightforward to estimate, however there are some important shortcomings.

- One undesirable property of the LPM is that, if we plug in certain combinations of values for the independent variables into (3.2), we can get predictions either less than zero or greater than one. Of course a probability by definition falls within the (0,1) interval, so predictions outside this range are meaningless and somewhat embarrassing. This is not an unusual result; for instance, based on the above LPM results, there are 61 observations for which the predicted probability is larger than one and 81 observations for which the predicted probability is less than zero. That is, 16 per cent of the predictions fall outside the (0,1) interval in this application (see Figure 1 in the appendix, and the summary statistics for the predictions reported below the table).
- Angrist and Pischke (p.103): "...[linear regression] may generate fitted values outside the LDV boundaries. This fact bothers some researchers and has generated a lot of bad press for the linear probability model."
- A related problem is that, conceptually, it does not make sense to say that a probability is **linearly** related to a continuous independent variable for all possible values. If it were, then continually increasing this explanatory variable would eventually drive  $P(y = 1|x)$  above one or below zero. For example, the model above predicts that an increase in parental wealth by 1 unit increases the probability of going to a PUA school by about 1 percentage point. This may seem reasonable for families with average levels of wealth, however in very rich or very poor families the wealth effect is probably smaller. In fact, when taken to the extreme our model implies that a hundred-fold increase in wealth increases the probability of going to a PUA by more than 1 which, of course, is impossible (the wealth variable ranges from 0.072 to 82 in the data, so such an comparison is not

unrealistic).

- A third problem with the LPM - arguably less serious than those above - is that the residual is heteroskedastic by definition. Why is this? Because  $y$  takes the value of 1 or 0, the residuals in equation (3.1) can take only two values, conditional on  $x$ :  $1 - \beta x$  and  $-\beta x$ . Further, the respective probabilities of these events are  $\beta x$  and  $1 - \beta x$ . Hence,

$$\begin{aligned} \text{var}(u|\mathbf{x}) &= \Pr(y = 1|\mathbf{x}) [1 - \mathbf{x}\beta]^2 \\ &\quad + \Pr(y = 0|\mathbf{x}) [-\mathbf{x}\beta]^2 \\ &= \mathbf{x}\beta [1 - \mathbf{x}\beta]^2 + (1 - \mathbf{x}\beta) [-\mathbf{x}\beta]^2 \\ &= \mathbf{x}\beta [1 - \mathbf{x}\beta], \end{aligned}$$

which clearly varies with the explanatory variables  $\mathbf{x}$ . The OLS estimator is still unbiased, but the conventional formula for estimating the standard errors, and hence the t-values, will be wrong. The easiest way of solving this problem is to obtain estimates of the standard errors that are robust to heteroskedasticity.

- EXAMPLE continued: Appendix - LPM with robust standard errors, Table 1b; compare to LPM with non-robust standard errors (Table 1a).
- A fourth and related problem is that, because the residual can only take two values, it cannot be normally distributed. The problem of non-normality means that OLS point estimates are unbiased but its violation does mean that inference in small samples cannot be based on the usual suite of normality-based distributions such as the  $t$  test.

Summarising:

- The LPM can be useful as a first step in the analysis of binary choices, but awkward issues arise if we want to argue that we are modelling a probability.

- As we shall see next, probit and logit solve these particular problems. Nowadays, these are just as easy to implement as LPM/OLS - but they are less straightforward to interpret.
- However, LPM remains a reasonably popular modelling framework (see e.g. Miguel, Satyanath and Sergenti, JPE, 2004), because certain econometric problems are easier to address within the LPM framework than with probits and logits.
- If, for whatever reason, we use the LPM, it is important to recognise that it tends to give better estimates of the partial effects on the response probability near the centre of the distribution of  $\mathbf{x}\boldsymbol{\beta}$  than at extreme values (i.e. close to 0 and 1). The LPM graph in the appendix illustrates this (Figure 1).

#### 4. Logit and Probit Models for Binary Response

The two main problems with the LPM were: nonsense predictions are possible (there is nothing to bind the value of Y to the (0,1) range); and linearity doesn't make much sense conceptually.

To address these problems we abandon the LPM and thus the OLS approach to estimating binary response models. Consider instead a class of binary response models of the form

$$\begin{aligned}\Pr(y = 1|\mathbf{x}) &= G(\beta_1 + \beta_2x_2 + \dots + \beta_Kx_K) \\ \Pr(y = 1|\mathbf{x}) &= G(\mathbf{x}\boldsymbol{\beta}),\end{aligned}\tag{4.1}$$

where  $G$  is a function taking on values strictly between zero and one:  $0 < G(z) < 1$ , for all real numbers  $z$ . The model (4.1) is often referred to in general terms as an **index model**, because  $\Pr(y = 1|x)$  is a function of the vector  $x$  only through the **index**

$$\mathbf{x}\boldsymbol{\beta} = \beta_1 + \beta_2x_2 + \dots + \beta_kx_k,$$

which is simply a scalar. Notice that  $0 < G(\mathbf{x}\boldsymbol{\beta}) < 1$  ensures that the estimated response probabilities



are strictly between zero and one, which thus addresses the main worries of using LPM.  $G$  is usually a **cumulative density function** (cdf), monotonically increasing in the index  $z$  (i.e.  $\mathbf{x}\boldsymbol{\beta}$ ), with

$$\Pr(y = 1|\mathbf{x}) \rightarrow 1 \text{ as } \mathbf{x}\boldsymbol{\beta} \rightarrow \infty$$

$$\Pr(y = 1|\mathbf{x}) \rightarrow 0 \text{ as } \mathbf{x}\boldsymbol{\beta} \rightarrow -\infty.$$

It follows that  $G$  must be a non-linear function, and hence we cannot use OLS.

Various non-linear functions for  $G$  have been suggested in the literature. By far the most common ones are the logistic distribution, yielding the **logit** model, and the standard normal distribution, yielding the **probit** model. In the logit model,

$$G(\mathbf{x}\boldsymbol{\beta}) = \frac{\exp(\mathbf{x}\boldsymbol{\beta})}{1 + \exp(\mathbf{x}\boldsymbol{\beta})} = \Lambda(\mathbf{x}\boldsymbol{\beta}),$$

which is between zero and one for all values of  $\mathbf{x}\boldsymbol{\beta}$  (recall that  $\mathbf{x}\boldsymbol{\beta}$  is a scalar). This is the cumulative distribution function (CDF) for a logistic variable. In the probit model,  $G$  is the standard normal CDF, expressed as an integral:

$$G(\mathbf{x}\boldsymbol{\beta}) = \Phi(\mathbf{x}\boldsymbol{\beta}) \equiv \int_{-\infty}^{\mathbf{x}\boldsymbol{\beta}} \phi(v) dv,$$

where

$$\phi(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right),$$

is the standard normal density. This choice of  $G$  also ensures that the probability of 'success' is strictly between zero and one for all values of the parameters and the explanatory variables.

EXAMPLE: See graphs in Figure 2, appendix.

The logit and probit functions are both increasing in  $\mathbf{x}\boldsymbol{\beta}$ . Both functions increase relatively quickly at  $\mathbf{x}\boldsymbol{\beta} = 0$ , while the effect on  $G$  at extreme values of  $\mathbf{x}\boldsymbol{\beta}$  tends to zero. The latter result ensures that the **partial effects** of changes in explanatory variables are **not constant**, a concern we had with the LPM. Also notice that the standard normal CDF has a shape very similar to of the logistic CDF, suggesting

that it doesn't much matter which one of the two we choose to use in our analysis. I will come back to this point later.

#### 4.1. A latent variable framework

As we have seen, the probit and logit models resolve some of the problems with the LPM model. The key, really, is the specification

$$\Pr(y = 1|x) = G(\mathbf{x}\boldsymbol{\beta}),$$

where  $G$  is the cdf for either the standard normal or the logistic distribution, because with any of these models we have a functional form that is easier to defend than the linear model. This, essentially, is how Wooldridge motivates the use of these models.

The traditional way of introducing probits and logits in econometrics, however, is not as a response to a functional form problem. Instead, probits and logits are traditionally viewed as models suitable for estimating parameters of interest when the dependent variable is not fully observed. Let's have a look at this perspective.

Let  $y^*$  be a continuous variable that we do not observe - a **latent variable** - and assume  $y^*$  is determined by the model

$$\begin{aligned} y^* &= \beta_1 + \beta_2 x_2 + \dots + \beta_K x_K + e \\ &= \mathbf{x}\boldsymbol{\beta} + e, \end{aligned} \tag{4.2}$$

where  $e$  is a residual, assumed uncorrelated with  $\mathbf{x}$  (i.e.  $\mathbf{x}$  is not endogenous). While we do not observe  $y^*$ , we do observe the discrete choice made by the individual, according to the following choice rule:

$$\begin{aligned} y &= 1 \text{ if } y^* > 0 \\ y &= 0 \text{ if } y^* \leq 0. \end{aligned}$$

Why is  $y^*$  unobserved? Think about  $y^*$  as representing net utility of, say, buying a car. The individual undertakes a cost-benefit analysis and decides to purchase the car if the net utility is positive. We do not observe (because we cannot measure) the 'amount' of net utility; all we observe is the actual outcome of whether or not the individual does buy a car. (If we had data on  $y^*$  we could estimate the model (9.2) with OLS as usual.)

Now, we want to model the probability that a 'positive' choice is made (e.g. buying, as distinct from not buying, a car). Assuming that  $e$  follows a logistic distribution<sup>2</sup>,

$$\begin{aligned}\lambda(e) &= \frac{\exp(-e)}{(1 + \exp(-e))^2} \text{ (density),} \\ \Lambda(e) &= \frac{\exp(e)}{1 + \exp(e)} \text{ (CDF),}\end{aligned}$$

it follows that

$$\begin{aligned}\Pr(y = 1|x) &= \Pr(y^* > 0|x) \\ &= \Pr(\mathbf{x}\boldsymbol{\beta} + e > 0|x) \\ &= \Pr(e > -\mathbf{x}\boldsymbol{\beta}) \\ &= 1 - \Lambda(-\mathbf{x}\boldsymbol{\beta}) \text{ (integrate)} \\ &= \Lambda(\mathbf{x}\boldsymbol{\beta}) \text{ (exploit symmetry).}\end{aligned}$$

Notice that the last step here exploits the fact that the logistic distribution is symmetric, so that  $G(z) = 1 - G(-z)$  for all  $z$ . This equation is exactly the binary response model (4.1) for the logit model. This is how the binary response model can be derived from an underlying latent variable model.

We can follow the same route to derive the probit model. Assume  $e$  follows a standard normal distribution

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<sup>2</sup>Note that symmetry of the probability density function implies  $\lambda(e) = \frac{\exp(e)}{(1+\exp(e))^2} = \frac{\exp(-e)}{(1+\exp(-e))^2}$ . In some expositions you see  $\frac{\exp(e)}{(1+\exp(e))^2}$ , in others  $\frac{\exp(-e)}{(1+\exp(-e))^2}$ . Don't let this confuse you.

$$\begin{aligned}
\Pr(y = 1|\mathbf{x}) &= \Pr(y^* > 0|\mathbf{x}) \\
&= \Pr(\mathbf{x}\boldsymbol{\beta} + e > 0|\mathbf{x}) \\
&= \Pr(e > -\mathbf{x}\boldsymbol{\beta}) \\
&= 1 - N\left(-\frac{\mathbf{x}\boldsymbol{\beta}}{\sigma}\right) \text{ (integrate)} \\
&= \Phi(\mathbf{x}\boldsymbol{\beta}),
\end{aligned}$$

where again we exploit symmetry and use  $\sigma = 1$  implied by the standard normal distribution. This is the binary response model (4.1) for the probit model.<sup>3</sup>

## 5. Estimation of logit and probit models

To estimate the LPM we can use OLS. Because of the non-linear nature of the probit and logit models (see graphs), however, linear estimators are not applicable for these. Instead we rely on **Maximum Likelihood (ML)** estimation. The principle of ML is very general and not confined to probit and logit models. Before turning the details of how ML is used to estimate probits and logits, here is an informal recap of ML.

### 5.1. Maximum Likelihood: Recap

- Suppose that, in the population, there is a variable  $w$  which is distributed according to some distribution  $f(w; \boldsymbol{\theta})$ , where  $\boldsymbol{\theta}$  is a vector of unknown parameters.
- Suppose we have a random sample  $\{w_1, w_2, \dots, w_N\}$  drawn from the population distribution  $f(w; \boldsymbol{\theta})$  where  $\boldsymbol{\theta}$  is unknown.
- Our objective is to estimate  $\boldsymbol{\theta}$ . Our sample is more likely to have come from a population charac-

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<sup>3</sup>The assumption that  $\sigma = 1$  may appear restrictive. In fact, this is a necessary normalisation, because we cannot estimate  $\sigma$  by means of a binary response model.

terised by one particular set of parameter values, say  $\tilde{\theta}$ , than from another set of parameter values, say  $\check{\theta}$ .

- The maximum likelihood estimate (MLE) of  $\theta$  is simply the particular vector  $\hat{\theta}^{ML}$  that gives the **greatest likelihood** (or, if you prefer, probability) of observing the sample  $\{w_1, w_2, \dots, w_N\}$ .
- **Random sampling** (an assumption) implies that  $w_1, w_2, \dots, w_N$  are independent of each other, hence the likelihood of observing  $\{w_1, w_2, \dots, w_N\}$  (i.e. the sample) is simply

$$L(\theta; w_1, w_2, \dots, w_N) = f(w_1; \theta) f(w_2; \theta) \cdot \dots \cdot f(w_N; \theta),$$

or, in more compact notation,

$$L(\theta; w_1, w_2, \dots, w_N) = \prod_{i=1}^N f(w_i; \theta).$$

i.e. the **product** of the individual likelihoods. The equation just defined is a function of  $\theta$ : for some values of  $\theta$  the resulting  $L$  will be relatively high while for other values of  $\theta$  it will be low. This is why we refer to equations of this form as **likelihood functions**.

- The value of  $\theta$  that gives the maximum value of the likelihood function is the maximum likelihood estimate of  $\theta$ .
- For computational reasons it is much more convenient to work with the **log-likelihood function**:

$$\ln L(\theta; w_1, w_2, \dots, w_N) = \sum_{i=1}^N \ln f(w_i; \theta).$$

The value of  $\theta$  that gives the maximum value of the log likelihood function is the  $\hat{\theta}^{ML}$ .

## 5.2. Maximum likelihood estimation of logit and probit models

- We now return to the logit and probit models. How can ML be used to estimate the parameters of interest in these models, i.e.  $\beta$ ? Assume that we have random sample of size  $N$ . The ML estimate of  $\beta$  is the particular vector  $\hat{\beta}^{ML}$  that gives the greatest likelihood of observing the sample  $\{y_1, y_2, \dots, y_N\}$ , conditional on the explanatory variables  $\mathbf{x}$ .
- By assumption, the probability of observing  $y_i = 1$  is  $G(\mathbf{x}_i\beta)$  while the probability of observing  $y_i = 0$  is  $1 - G(\mathbf{x}_i\beta)$ . It follows that the probability of observing the entire sample is

$$L(y|\mathbf{x}; \beta) = \prod_{i \in l} G(\mathbf{x}_i\beta) \prod_{i \in m} [1 - G(\mathbf{x}_i\beta)],$$

where  $l$  refers to the observations for which  $y = 1$  and  $m$  to the observations for which  $y = 0$ .

- We can rewrite this as

$$L(y|\mathbf{x}; \beta) = \prod_{i=1}^N G(\mathbf{x}_i\beta)^{y_i} [1 - G(\mathbf{x}_i\beta)]^{(1-y_i)},$$

because when  $y = 1$  we get  $G(\mathbf{x}_i\beta)$  and when  $y = 0$  we get  $[1 - G(\mathbf{x}_i\beta)]$ .

- The **log** likelihood for the sample is

$$\ln L(y|\mathbf{x}; \beta) = \sum_{i=1}^N \{y_i \ln G(\mathbf{x}_i\beta) + (1 - y_i) \ln [1 - G(\mathbf{x}_i\beta)]\}.$$

The MLE of  $\beta$  **maximises** this log likelihood function.

- If  $G$  is the logistic CDF then we obtain the logit likelihood:

$$\begin{aligned} \ln L(y|\mathbf{x}; \beta) &= \sum_{i=1}^N \{y_i \ln \Lambda(\mathbf{x}_i\beta) + (1 - y_i) \ln [1 - \Lambda(\mathbf{x}_i\beta)]\} \\ \ln L(y|\mathbf{x}; \beta) &= \sum_{i=1}^N \left\{ y_i \ln \left( \frac{\exp(\mathbf{x}_i\beta)}{1 + \exp(\mathbf{x}_i\beta)} \right) + (1 - y_i) \ln \left( \frac{1}{1 + \exp(\mathbf{x}_i\beta)} \right) \right\}, \end{aligned}$$

which simplifies to

$$\ln L(y|\mathbf{x}; \boldsymbol{\beta}) = \sum_{i=1}^N \{y_i [\mathbf{x}_i \boldsymbol{\beta} - \ln(1 + \exp(\mathbf{x}_i \boldsymbol{\beta}))] - (1 - y_i) \ln(1 + \exp(\mathbf{x}_i \boldsymbol{\beta}))\}.$$

- If  $G$  is the standard normal CDF we get the probit estimator:

$$\ln L(y|\mathbf{x}; \boldsymbol{\beta}) = \sum_{i=1}^N \{y_i \ln \Phi(\mathbf{x}_i \boldsymbol{\beta}) + (1 - y_i) \ln [1 - \Phi(\mathbf{x}_i \boldsymbol{\beta})]\}.$$

### How maximise the log likelihood function?

- Sample log likelihood:

$$\ln L(y|\mathbf{x}; \boldsymbol{\beta}) = \sum_{i=1}^N \{y_i \ln G(\mathbf{x}_i \boldsymbol{\beta}) + (1 - y_i) \ln [1 - G(\mathbf{x}_i \boldsymbol{\beta})]\}.$$

- Because the objective is to maximise the log likelihood function with respect to the parameters in the vector  $\boldsymbol{\beta}$ , it must be that, at the maximum, the following  $K$  first order conditions will hold:

$$\sum_{i=1}^N \left\{ y_i \frac{g(\mathbf{x}_i \boldsymbol{\beta})}{G(\mathbf{x}_i \boldsymbol{\beta})} + (1 - y_i) \frac{g(\mathbf{x}_i \boldsymbol{\beta})}{[1 - G(\mathbf{x}_i \boldsymbol{\beta})]} \right\} \mathbf{x}_i = 0.$$

1 x 1

1 x K

- It is typically not possible to solve analytically for  $\boldsymbol{\beta}$  here. Instead, to obtain parameter estimates, we rely on some sophisticated iterative 'trial and error' technique. There are lots of **algorithms** that can be used, but we will not study these here. The most common ones are based on first and sometimes second derivatives of the log likelihood function. Think of a blind man walking up a hill, and whose only knowledge of the hill comes from what passes under his feet. Provided the hill is strictly concave, the man should have no trouble finding the top. Fortunately the log likelihood functions for logit and probit are concave, but this is not always the case for other models.

EXAMPLE: Appendix, Tables 2-3.

## 6. Interpreting logit and probit results

In most cases the main goal is to determine the effects on the response probability  $\Pr(y = 1|x)$  resulting from a change in one of the explanatory variables, say  $x_j$ .

### 6.1. Case I: The explanatory variable is continuous.

- In linear models the marginal effect of a unit change in some explanatory variable on the dependent variable is simply the associated coefficient on the relevant explanatory variable.
- However, for logit and probit models obtaining measures of the marginal effect is more complicated (which should come as no surprise, as these models are non-linear). When  $x_j$  is a continuous variable, its partial effect on  $\Pr(y = 1|x)$  is obtained from the partial derivative:

$$\begin{aligned}\frac{\partial \Pr(y = 1|x)}{\partial x_j} &= \frac{\partial G(\mathbf{x}\boldsymbol{\beta})}{\partial x_j} \\ &= g(\mathbf{x}\boldsymbol{\beta}) \cdot \beta_j,\end{aligned}$$

where

$$g(z) \equiv \frac{dG(z)}{dz}$$

is the **probability density function** associated with  $G$ .

- Because the density function is non-negative, the partial effect of  $x_j$  will always have **the same sign** as  $\beta_j$ .
- Notice that the partial effect depends on  $g(\mathbf{x}\boldsymbol{\beta})$ ; i.e. for different values of  $x_1, x_2, \dots, x_k$  the partial effect will be different.
- Can you see at what values of  $\mathbf{x}\boldsymbol{\beta}$  the partial (or marginal) effect will be relatively small/large? See graphs of the standard normal and the logistic CDFs in handout.



EXAMPLE: Suppose we use the Indian data introduced above to estimate a probit modelling the probability that a child goes to a private unaided school as a function of the child's ability, measured by the score on the Raven's test. For simplicity, abstract from other explanatory variables. Our model is thus:

$$\Pr(pua = 1|sraven) = \Phi(\beta_0 + \beta_1 sraven).$$

The probit results are

	coef.	t-value
$\beta_0$	-1.82	12.84
$\beta_1$	0.050	11.76

Since the coefficient on *sraven* is positive, we know that the marginal effect must be positive. Treating *sraven* as a continuous variable, it follows that the marginal effect is equal to

$$\begin{aligned} \frac{\partial \Pr(pua = 1|sraven)}{\partial sraven} &= \phi(\beta_0 + \beta_1 \cdot sraven) \beta_1 \\ &= \phi(-1.82 + 0.05 \cdot sraven) 0.05, \end{aligned}$$

where  $\phi(\cdot)$  is the standard normal density function:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2).$$

We see straight away that the marginal effect depends on the level of *sraven*. We see from the summary statistics that the mean value of *sraven* is about 31, so let's evaluate the marginal effect at *sraven* = 31:

$$\begin{aligned} &\frac{\partial \Pr(pua = 1|sraven = 31)}{\partial sraven} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-(-1.82 + 0.05 \cdot 31)^2/2\right) 0.05 \\ &= 0.019, \end{aligned}$$

Evaluated at the mean of *sraven*, we see that the results imply that an increase in *sraven* by one unit raises the probability of going to a private school by about two percentage points. At lower levels of *sraven*, the marginal effect is smaller:

$$\begin{aligned} & \frac{\partial \Pr(pua = 1 | sraven = 15)}{\partial sraven} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-(-1.82 + 0.05 \cdot 20)^2 / 2\right) 0.05 \\ &= 0.011. \end{aligned}$$

Of course, the fact that the marginal effect is smaller at lower levels reflects the non-linearity of the probit model (again: see graphs in Figure 2 in handout).

STUDENT EXERCISE: Now consider **logit**:

$$\Pr(pua = 1 | sraven) = \Lambda(\beta_0 + \beta_1 sraven),$$

$$\Lambda(z) = \frac{\exp(z)}{1 + \exp(z)}.$$

The logit results are

	coef.	t-value
$\beta_0$	-3.07	12.00
$\beta_1$	0.084	11.20

Task: Calculate and interpret the marginal effect. Compare the result to the probit marginal effect.

The Stata command 'mfx compute' can be used to obtain marginal effects, with standard errors, after logit and probit models.

## 6.2. Case II: The explanatory variable is discrete.

If  $x_j$  is a discrete variable then we should not rely on calculus in evaluating the effect on the response probability. To keep things simple, suppose  $x_2$  is binary. In this case the partial effect from changing  $x_2$

from zero to one, holding all other variables fixed, is

$$G(\beta_1 + \beta_2 \cdot 1 + \dots + \beta_K x_K) - G(\beta_1 + \beta_2 \cdot 0 + \dots + \beta_K x_K).$$

Again this depends on all the values of the other explanatory variables and the values of all the other coefficients.

Again, knowing the **sign** of  $\beta_2$  is sufficient for determining whether the effect is positive or not, but to find the **magnitude** of the effect we have to use the formula above.

The Stata command 'mfx compute' can spot dummy explanatory variables. In such a case it will use the above formula for estimating the partial effect.

### 6.3. Case III: Non-linear explanatory variables.

Suppose the model is

$$\Pr(y = 1|x) = G(\beta_1 + \beta_2 x_2 + \beta_2 x_3 + \beta_{22} x_2^2),$$

where  $x_2^2$  is a continuous variable. What is the marginal effect of  $x_2$  on the response probability?

## 7. Probit and Logit: Testing Hypotheses

### 7.1. Inference based on the log likelihood function

- We have already discussed how the ML estimates of the parameters are those that maximise the likelihood of observing the sample. It must then be that **all other parameter values** - which, by definition, are not the ML estimates - will result in a **lower** (worse) log likelihood value.
- Now let's revisit our Indian dataset and investigate what happens to the log likelihood value if we change the value of the coefficient on *sraven*. See Figure 3 in the handout.
- As expected, values of `b_raven` not equal to 0.03 produce lower log likelihood values.
- Is it important **how much** the log L falls as a result of moving `b_raven` a given distance away

from the ML estimate of 0.03? Yes, very important, because this, essentially, **is** the general basis for our inference. Think about the log likelihood ratio test.

- The **log likelihood ratio** test is defined as two times the difference in two log likelihood values:

$$LR = 2 (\ln L_{ur} - \ln L_r),$$

where  $\ln L_{ur}$  is the log likelihood value for the **unrestricted** model and  $\ln L_r$  is the log likelihood value for the **restricted** model.  $LR$  follows a chi-squared distribution with  $q$  degrees of freedom under  $H_0$ , where  $q$  is the number of restrictions.

- Suppose now I want to test the following null hypothesis:

$$H_0 : b\_sraven = 0.$$

Looking at Figure 3, I see that

$$\ln L_r \simeq -356,$$

(i.e. this is the log likelihood value associated with  $b\_sraven = 0$ ) and I know from the regression output (or from the graph) that

$$\ln L_{ur} = -340.4.$$

Hence

$$LR = 2 (-340.4 + 356) = 31.2.$$

To test  $H_0$  at the 5% level we use as our critical value the 95<sup>th</sup> percentile in the  $\chi_q^2$  distribution. With  $q = 1$  (because there is only one restriction here) the critical value is 3.84, so I firmly reject the null hypothesis at the 5% level. If you want a specific p-value, we can type

$$chiprob(1, 31.2)$$

in Stata which is equal to 0.00000002. We can thus reject the null at any conventional level of significance.

- Key point: It is the **large fall** in the log L resulting from imposing  $b\_sraven = 0$  that enables us to reject the null hypothesis. Had the log likelihood function been flatter in  $b\_sraven$ , we might not have been able to reject the null hypothesis.
- The log likelihood ratio is often used to test whether a sub-set of the explanatory variables can be omitted from the model. Again the idea is that, because ML maximizes the log likelihood function, dropping variables will lead to a lower log likelihood value (this is similar to the result that the R-squared falls when variables are dropped from an OLS regression). The question is whether the fall in the log likelihood is large enough to conclude that the dropped variables are important. The likelihood ratio statistic:

$$LR = 2 (\ln L_{ur} - \ln L_r),$$

where  $\ln L_{ur}$  is the log likelihood value for the unrestricted model, e.g.

$$G(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4),$$

and  $\ln L_r$  is the value for the restricted model, e.g.

$$G(\beta_0 + \beta_1 x_1).$$

So estimate these two models and compare the two log likelihood values. We can obtain p-values directly in Stata by using the command

$$chprob(q, LR).$$

What's  $q$  in this case?

- Example in appendix, Table 8.
- In Table 2 in the appendix, how should we interpret the information in 'LR chi2(9)'?

## 7.2. Standard errors for parameters

In linear models the covariance matrix is given by  $\sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$  which is straightforward to estimate. In non-linear models, however, such as the probit and the logit, deriving formulas for the covariance matrix, and hence the standard errors, is more complicated. The conventional estimator for the covariance matrix is based on the inverse of the negative Hessian:

$$\text{Var}(\hat{\boldsymbol{\beta}}^{ML}) = -H^{-1}, \quad (7.1)$$

where the Hessian is the matrix of second order derivatives of the log likelihood function:

$$H(\boldsymbol{\beta}) = \frac{\partial^2 \ln L(y|\mathbf{x}_i; \hat{\boldsymbol{\beta}}^{ML})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'}$$

Note that provided the log likelihood function is concave, the second derivative is negative which ensures that the variance is positive.

- This is somewhat intuitive. Note that the second derivative of the log likelihood function with respect to  $\boldsymbol{\beta}$  (evaluated at  $\hat{\boldsymbol{\beta}}^{ML}$ ) measures the **curvature** of the log likelihood function - and so variance formula (7.1) says that **the more curvature, the lower is the variance**.
- Recall that how big is the quantitative fall in the log L as a result of imposing other parameter values than the ML estimate, is central for our inference if we use the log likelihood ratio test. Clearly curvature plays a central role here: with little curvature you have to move the parameter value for `b_sraven` a long way away from its ML estimate before the LR test rejects, but with a lot of curvature you will not have to move far. So you see the variance and the LR test are very closely related.

- Sometimes you see 'robust' standard errors reported - these are obtained from the 'sandwich' formula:

$$Var(\hat{\beta}^{ML}) = [-H(\hat{\beta}^{ML})]^{-1} Var[s(\hat{\beta}^{ML})] [-H(\hat{\beta}^{ML})]^{-1},$$

where

$$s(\hat{\beta}^{ML}) \equiv \frac{\partial \ln L(y|\mathbf{x}; \hat{\beta}^{ML})}{\partial \hat{\beta}^{ML}}$$

is the **gradient** vector, or **score** vector.

### 7.3. Standard errors for marginal effects

- Once we have estimated the variance matrix, we can calculate standard errors by taking the square root of the diagonal elements of the covariance matrix, and subsequently obtain t-values and confidence intervals in the usual ways.
- We can also calculate standard errors for the **marginal effects** (recall these are non-linear functions of the parameters). The Stata mfx command does this for us using the **delta method**, which involves transforming the standard errors of  $\hat{\beta}^{ML}$  into standard errors of  $\frac{\partial \Pr(y=1|\mathbf{x})}{\partial x_j}$  by means of a Taylor series approximation. Here is how it works:

- Our goal is to estimate the standard error of the marginal effect  $\frac{\partial \Pr(y=1|\mathbf{x})}{\partial x_j}$ . Define

$$\frac{\partial \Pr(y=1|\mathbf{x})}{\partial x_j} \equiv h(\beta),$$

making it explicit that the marginal effect is a function of the parameters  $\beta$ . We have obtained an estimate of  $\beta$ , denoted  $\hat{\beta}^{ML}$ . We have also estimated the covariance matrix  $Var(\hat{\beta}^{ML})$ .

We now need to obtain  $Var(\hat{\gamma}_j^{ML})$ .

- Now define  $\hat{\gamma}_j^{MLE} = h(\hat{\beta}^{MLE})$ , and then take a Taylor series approximation of  $\hat{\gamma}_j^{MLE} =$

$h(\hat{\beta}^{MLE})$  around the true value  $\beta$ :

$$\hat{\gamma}_j^{MLE} \simeq \gamma_j + \sum_{i=1}^K \frac{\partial h}{\partial \beta_i} (\hat{\beta}_i^{MLE} - \beta_i).$$

In matrix notation

$$\hat{\gamma}^{MLE} - \gamma \simeq \Psi (\hat{\beta}^{MLE} - \beta), \tag{7.2}$$

where

$$\Psi = \begin{bmatrix} \frac{\partial h_1}{\partial \beta_1} & \frac{\partial h_1}{\partial \beta_2} & \cdots & \frac{\partial h_1}{\partial \beta_K} \\ \frac{\partial h_2}{\partial \beta_1} & \frac{\partial h_2}{\partial \beta_2} & \cdots & \frac{\partial h_2}{\partial \beta_K} \\ \cdots & & & \\ \frac{\partial h_J}{\partial \beta_1} & \cdots & & \frac{\partial h_J}{\partial \beta_K} \end{bmatrix}$$

is a  $J \times K$  matrix of derivatives. Post-multiply (7.2) by the transpose of (7.2), and take expectations, and you get the variance matrix for the marginal effects:

$$Var(\hat{\gamma}^{MLE}) = \Psi Var(\hat{\beta}^{MLE}) \Psi'.$$

- A good discussion of the delta method is provided by Deaton (1997) 'The Analysis of Household Surveys: A Microeconomic Approach to Development Policy', pp. 128-129 (not required reading).

## 8. Model Diagnostics: Probit and Logit

### 8.1. Functional form

Remember that we abandoned the LPM essentially because we didn't find the linear functional form appealing. Instead, we wrote the general specification for a binary choice model as

$$\Pr(y = 1|\mathbf{x}) = G(\beta_1 + \beta_2 x_2 + \dots + \beta_K x_K),$$



where for  $G(\cdot)$  we considered the CDFs of the normal and logistic distributions, yielding probit and logit, respectively.

- Of course, there is no guarantee that normal or logistic CDFs are always appropriate functional forms.
- Neither is there a reason why the linear specification  $\beta_1 + \beta_2 x_2 + \dots + \beta_K x_K$  of the index going **into**  $G(\cdot)$  is correct.

So there are two sources of mis-specification of the binary choice model above:

- The **function**  $G(\cdot)$  - usually normal or logistic - could be wrong.
- The **argument** of the function  $G(\cdot)$  could have the wrong functional form (e.g. perhaps we specify  $\mathbf{x}\boldsymbol{\beta} = \beta_1 + \beta_2 x_2$ , when in fact the correct specification is  $\beta_1 + \beta_2 x_2 + \beta_{22} x_2^2$ ).

We need some methods that will enable us to detect a mis-specified functional form. I will introduce two simple methods that are useful to this end. Suppose that we have just estimated the following probit model

$$\Pr(y = 1|\mathbf{x}) = \Phi(\beta_1 + \beta_2 x_2 + \dots + \beta_K x_K)$$

(if your model is logit, just replace  $\Phi(\cdot)$  by  $\Lambda(\cdot)$  throughout). Importantly, my null hypothesis is that this is a **correct** specification - i.e.  $\Phi$  is the right function, and  $\beta_1 + \beta_2 x_2 + \dots + \beta_K x_K$  is the right functional form of the argument (index) of  $\Phi(\cdot)$ .

**A semi-parametric and 'informal' method for testing for mis-specification.** Given our parameter estimates  $\hat{\beta}_1, \dots, \hat{\beta}_K$ , we begin by calculating the index  $\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE}$ :

$$\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE} = \hat{\beta}_1 + \hat{\beta}_2 x_2 + \dots + \hat{\beta}_K x_K$$

(this can be done in Stata using 'predict', combined with the option xb). Under the null hypothesis that this is the right form of the index, and that  $\Phi(\cdot)$  is the right function, we have

$$\begin{aligned} E(y|x) &= \Phi\left(\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE}\right), \\ y &= \Phi\left(\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE}\right) + v \end{aligned}$$

using  $\Pr(y = 1|\mathbf{x}) = E(y|\mathbf{x})$  and  $y = E(y|\mathbf{x}) + v$ . It follows that if we regress  $y$  on  $\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE}$  allowing for a **general** (i.e. not necessarily normal) functional form  $f(\cdot)$ :

$$y = f\left(\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE}\right) + v,$$

the estimated function  $\hat{f}$  should closely track the cumulative density function for the normal distribution, if the null hypothesis is true.

To allow for a flexible functional form for  $f(\cdot)$  we can use some **semi-parametric method**. One such method which is both common and easy to implement (in Stata) is **lowess smoothing**.

Very informally, lowess smoothing estimates  $f(\cdot)$  as follows: For each observation in the data, calculate the expected value of  $Y_i$  conditional on  $X_i$  from a regression that a) uses only observations for which  $X$  is 'close' to  $X_i$ ; and b) uses weights which are determined by how close each observation  $X_j$  is to  $X_i$  (closer to  $X_i =$  higher weight). Once this has been done for all observations in the data, we can plot the  $N$  estimates of the conditional expectation of  $Y$  given  $X$  on the vertical axis and  $X$  on the horizontal axis. This plot is our estimate of the function  $f(\cdot)$ .

- See appendix, Figure 4, for an illustration.

**Example.** Consider the probit specification shown in Table 2 in the appendix. Figure 5 shows the estimated  $f(\cdot)$  based on the Stata command lowess. Although the curvature of  $f(\cdot)$  is somewhat less pronounced than that of  $\Phi(\cdot)$  at low levels of the index  $\hat{\beta}x$ , on the whole it seems the functional form of  $\Phi(\cdot)$  is not that bad.

Formal testing procedures drawing on these techniques have been developed, but unfortunately these are quite hard to implement and I will not review these here<sup>4</sup> The procedure just outlined is a simple informal way by which 'eye-balling' the data should tell us if the functional form we've assumed appears broadly consistent with the data. If you want to learn more about how these methods can be used in applied work, see Section 3.3 in Martins (2001) and Section 7 in Pagan (2002).

**A parametric and 'formal' method for testing for mis-specification.** The starting point is the same as in the previous section, i.e. based on the probit model we have the estimated index  $\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE}$ :

$$\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE} = \hat{\beta}_1 + \hat{\beta}_2 x_2 + \dots + \hat{\beta}_K x_K.$$

Under the null hypothesis,

$$\Pr(y = 1|\mathbf{x}) = \Phi(\mathbf{x}\boldsymbol{\beta}).$$

Now consider the alternative specification

$$\Pr(y = 1|\mathbf{x}) = \Phi\left(\mathbf{x}\boldsymbol{\beta} + \gamma_1 (\mathbf{x}\boldsymbol{\beta})^2 + \gamma_2 (\mathbf{x}\boldsymbol{\beta})^3\right).$$

Under the null hypothesis that our probit model is **correctly specified**, we have:  $\gamma_1 = \gamma_2 = 0$  (yes?).

If this hypothesis is rejected, then we have reason to believe our initial probit model is mis-specified. We can test this hypothesis by running a probit in which  $y$  is the dependent variable and  $\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE}$ ,  $(\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE})^2$  and  $(\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE})^3$  are the 'explanatory' variables (we need to impose a 'coefficient' on  $\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE}$  equal to one, and to ensure there is no constant in the model, but this is easily done in Stata). This procedure is discussed in Pagan and Vella (1989), Section 4.1 (see in particular p. S43).

- See appendix, Table 9.

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<sup>4</sup>Standard reference: Horowitz, J. L. 1993. "Semiparametric Estimation of a Work-trop Mode Choice Model," *Journal of Econometrics* 58, pp. 49-70.

Based on the two methods for assessing the adequacy of functional form, we would not reject the basic model. Of course, for other applications the conclusion might be different. When this happens, what do we do? There are essentially two options:

- Change the functional form for  $G(\cdot)$ . It's possible to go semiparametric, which is flexible - but not easy.
- Change the functional form of what goes **into**  $G(\cdot)$ , i.e. the specification of the 'index'. It is quite possible allowing for higher order terms in the explanatory variables will solve the problem. This is certainly a more tractable approach.

## 8.2. Goodness of fit

In linear models where the dependent variable is continuous, we often rely on the R-squared as a measure of the goodness of fit of the model. If for some reason we use linear regression in a binary choice setting (i.e. LPM here), you will obviously get an estimate of the R-squared. However, you should probably not pay too much attention to this statistic. Why?

Recall:

$$R^2 = \frac{\text{var}(\hat{y})}{\text{var}(y)},$$

where  $\hat{y}$  denotes the predictions from the regression. But remember the main problem with LPM is that linearity is an unattractive feature of the model - both conceptually and in the sense that nonsense probability predictions may result. Consequently, we should not take the predictions of the LPM too seriously and so any measures of how 'good' these predictions are, is of limited interest.

The two most common alternative measures of goodness of fit for binary choice models are the **percent correctly predicted**, and the **pseudo R-squared**.

**Percent correctly predicted.** To obtain the percent correctly predicted we begin by computing the estimated probability that  $y_i$  equals one for each observation in the sample. For the probit model, for

instance, this is given by

$$Est. \Pr(y = 1|x) = \Phi\left(\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE}\right),$$

where *Est.* denotes 'estimated'. We then say that the **predicted outcome** of  $y_i$  is one if  $\Phi\left(\mathbf{x}\hat{\boldsymbol{\beta}}^{MLE}\right) > 0.50$  and zero otherwise. The percentages of times the predicted  $y_i$  matches the actual  $y_i$  is the per cent correctly specified. Note the difference between predicted outcome (which is binary, 0 or 1) and predicted probability (any number between 0 and 1).

The per cent correctly predicted is a useful measure in this context, but we need to be careful. Consider a case where out of 200 observations, 180 have  $y_i = 0$ . If, say, 150 of these are predicted to be zero we obtain 75% correct predictions, even if our model fails to predict any of the observations for which  $y = 1$  correctly. This is not an uncommon outcome in practice. For this reason, it is a good idea to report the percentages (or frequencies) correctly predicted **for each of the two outcomes**.

- Appendix, Table 7.
- Note: Hard to say a priori what makes up a 'satisfactory' percentage of correct predictions.

**Pseudo R-squared.** Various pseudo R-squared measures for binary response models have been developed. The most common one is

$$\tilde{R}^2 = 1 - \frac{\ln L_{ur}}{\ln L_r},$$

where  $\ln L_{ur}$  is the value of the log likelihood at the ML estimates (the 'unrestricted' model) and  $\ln L_r$  is the log likelihood value for a 'restricted' model in which the only 'explanatory' variable is a constant.

What is the logic of using this formula? Notice that if our explanatory variables have no explanatory power at all, then  $\ln L_r = \ln L_{ur}$  (why?). In this case we get  $\tilde{R}^2 = 0$ .

In contrast, if our model is doing very well indeed in predicting the actual observations of  $y$ , then the log likelihood value (of the unrestricted model) will approach zero from below, and hence  $\tilde{R}^2$  will tend to one. Why?

Recall that the log likelihood function is

$$\ln L(y|x_i; \beta) = \sum_{i=1}^N \{y_i \ln G(\mathbf{x}_i\beta) + (1 - y_i) \ln [1 - G(\mathbf{x}_i\beta)]\}.$$

A very good model will be such that  $G(\mathbf{x}_i\beta)$  will be very close to one for all observations for which  $y_i = 1$  and very close to zero for all observations for which  $y_i = 0$ . To illustrate the point, suppose  $G(\mathbf{x}_i\beta)$  is exactly one for all observations for which  $y_i = 1$  and exactly zero for all observations for which  $y_i = 0$  - i.e. the model predicts the dependent variable perfectly. In that extreme case, we have

$$\begin{aligned} \ln L(y|x_i; \beta) &= \sum_{i=1}^N \{y_i \ln 1 + (1 - y_i) \ln [1 - 0]\} \\ &= \sum_{i=1}^N \{y_i \cdot 0 + (1 - y_i) \cdot 0\} \\ &= 0, \end{aligned}$$

and so

$$\begin{aligned} \tilde{R}^2 &= 1 - \frac{0}{\ln L_r} \\ &= 1. \end{aligned}$$

Notice that  $\tilde{R}^2$  uses the same information as that underlying the log likelihood ratio test.

- See Table 2 in handout. Verify that the reported  $\tilde{R}^2$  is consistent with the LR test.

## 9. More topics: Heteroskedasticity & Endogeneity

### 9.1. Heteroskedasticity

A lot has been written about the problems posed by heteroskedasticity for the probit and logit models. You often hear statements to the effect that probit and logit estimates are **inconsistent** in the presence of heteroskedasticity. Greene (2003, p. 679) argues that this is a serious problem "because the probit model is most often used with microeconomic data, which are frequently heteroscedastic". What is the nature of the problem? Consider the following illustration, taken from Section 15.7.4 in Wooldridge (2002).

Start from a latent variable model with one explanatory variable  $x_{i1}$ :

$$y_i^* = \psi_0 + \psi_1 x_{i1} + u_i. \quad (9.1)$$

Suppose the residual  $u_i$  is heteroskedastic. Consider the following - admittedly very special and arguably peculiar - form of heteroskedasticity:

$$u_i \sim Normal(0, x_{i1}^2).$$

Recall that we do not observe  $y_i^*$  - all we observe is the binary dependent variable:

$$y_i = 1 \text{ if } y_i^* > 0$$

$$y_i = 0 \text{ if } y_i^* \leq 0.$$

Thus,

$$y_i = 1 \text{ if } \psi_0 + \psi_1 x_{i1} + u_i > 0.$$

What is the probability that  $y = 1$ ? We have

$$\begin{aligned} \Pr(y_i = 1|x_i) &= \Pr(y_i^* > 0|x_i) \\ &= \Pr(\psi_0 + \psi_1 x_{i1} + u_i > 0|x_i) \\ &= \Pr\left(\psi_0 + \psi_1 x_{i1} + \sqrt{x_{i1}^2} e_i > 0|x_i\right), \end{aligned}$$

where  $e_i$  follows a standard normal distribution (i.e. with mean zero and variance equal to one). Hence,

$$\begin{aligned} \Pr(y_i = 1|x_{i1}) &= \Pr\left(e_i > -\frac{1}{x_{i1}}(\psi_0 + \psi_1 x_{i1})\right) \\ &= 1 - \Phi\left(-\frac{1}{x_{i1}}(\psi_0 + \psi_1 x_{i1})\right) \text{ (integrate)} \\ &= \Phi\left(\frac{1}{x_{i1}}(\psi_0 + \psi_1 x_{i1})\right), \text{ (symmetry)} \\ &= \Phi\left(\psi_0 \frac{1}{x_{i1}} + \psi_1\right). \end{aligned}$$

We now see how the presence of heteroskedasticity radically has **altered the functional form** of the probit model. Given that the underlying latent model is

$$y_i^* = \psi_0 + \psi_1 x_{i1} + u_i,$$

we might be tempted to specify the probit model as

$$\Pr(y_i = 1|x_i) = \Phi(\psi_0 + \psi_1 x_{i1}),$$

but this would **not** be the correct specification. This is quite important. Think about the partial effect of  $x_{i1}$ . The correct specification is

$$\Pr(y_i = 1|x_i) = \Phi\left(\psi_0 \frac{1}{x_{i1}} + \psi_1\right),$$



hence the correct marginal effect is

$$\frac{\partial \Pr(y_i = 1|x_{i1})}{\partial x_{i1}} = \phi\left(\psi_0 \frac{1}{x_{i1}} + \psi_1\right) \left(-\psi_0 \left(\frac{1}{x_{i1}}\right)^2\right).$$

Remarkably, the sign of the marginal effect is the opposite of that of  $\psi_0$  - i.e. the constant in the latent variable model - and does not depend on the sign of  $\psi_1$  - the slope coefficient on  $x_{i1}$  in the latent variable model. It follows that if  $\psi_0$  and  $\psi_1$  are both positive, the marginal effect of  $x_{i1}$  on the probability of 'success' has the **opposite sign** to the marginal effect of  $x_{i1}$  on the latent dependent variable  $y_i^*$ .

Of course the latter result is driven by the specific form of heteroskedasticity considered here, and should not be viewed as a general result. The main point is that if the residual in the latent variable model is heteroskedastic this alters the functional form. Exactly how depends on the form of heteroskedasticity.

Now, suppose you were to specify (incorrectly) the probit as

$$\Pr(y_i = 1|x_i) = \Phi(\eta_0 + \eta_1 x_{i1}).$$

Do you think your coefficient on  $x_1$  would be a good estimate of the coefficient  $\psi_1$  in the latent variable model

$$y_i^* = \psi_0 + \psi_1 x_{i1} + u_i ?$$

Answer: no. And this is an example of how the presence of heteroskedasticity leads to "inconsistent estimates" of the parameters in the latent variable model.

How can we proceed if we believe heteroskedasticity is a problem? One possibility is to use Stata's **hetprobit** command, which estimates a generalized probit model:

$$\begin{aligned} y^* &= \beta_1 + \beta_2 x_2 + \dots + \beta_K x_K + e \\ y^* &= \mathbf{x}\boldsymbol{\beta} + e, \end{aligned} \tag{9.2}$$

where

$$\sigma_e^2 = [\exp(\mathbf{z}\boldsymbol{\gamma})]^2,$$

where  $\mathbf{z}$  is a vector of variables (not including a constant - since not identified) thought to affect the variance of  $e$ , and  $\boldsymbol{\gamma}$  is the corresponding vector of coefficients. We obtain

$$\begin{aligned}\Pr(y = 1|\mathbf{x}, \mathbf{z}) &= \Pr(y^* > 0|\mathbf{x}, \mathbf{z}) \\ &= \Pr(\mathbf{x}\boldsymbol{\beta} + e > 0|\mathbf{x}, \mathbf{z}) \\ &= \Pr(\mathbf{x}\boldsymbol{\beta} + \exp(\mathbf{z}\boldsymbol{\gamma})u > 0|\mathbf{x}, \mathbf{z}),\end{aligned}$$

where  $u$  follows a standard normal distribution (a normalization). Hence

$$\begin{aligned}\Pr(y = 1|\mathbf{x}, \mathbf{z}) &= \Pr\left(u > \frac{-\mathbf{x}\boldsymbol{\beta}}{\exp(\mathbf{z}\boldsymbol{\gamma})}\right) \\ \Pr(y = 1|\mathbf{x}, \mathbf{z}) &= 1 - N\left(-\frac{\mathbf{x}\boldsymbol{\beta}}{\exp(\mathbf{z}\boldsymbol{\gamma})}\right) \text{ (integrate)} \\ \Pr(y = 1|\mathbf{x}, \mathbf{z}) &= \Phi\left(\frac{\mathbf{x}\boldsymbol{\beta}}{\exp(\mathbf{z}\boldsymbol{\gamma})}\right).\end{aligned}$$

Of course, if a variable  $x_k$  is included in both  $\mathbf{x}$  and  $\mathbf{z}$ , the marginal effect is somewhat more involved:

$$\frac{\partial \Pr(y = 1|\mathbf{x}, \mathbf{z})}{\partial x_k} = \phi\left(\frac{\mathbf{x}\boldsymbol{\beta}}{\exp(\mathbf{z}\boldsymbol{\gamma})}\right) \left(\frac{\beta_k - (\mathbf{x}\boldsymbol{\beta})\gamma_k}{\exp(\mathbf{z}\boldsymbol{\gamma})}\right).$$

This shows that the sign of the marginal effect is not necessarily the same as the sign of  $\beta_k$ .

EXAMPLE: Heteroskedasticity in school choice probit. Appendix.

## 9.2. Endogeneous regressors

Reference: Wooldridge (2002), 15.7.2-3.

We have talked a lot earlier in this course about the problems posed by the explanatory variables being correlated with the residual, in the context of linear models. All the conceptual issues (e.g. reasons

for endogeneity, implications etc.) carry over to binary choice models (as well as other nonlinear models). Unfortunately, estimation and interpretation of probit or logit models with instrumental variables is not entirely straightforward. Clearly, one option would be to estimate a linear probability model by 2SLS. Recall that this is precisely what Miguel et al. (2004) do (computer exercise 1). As Wooldridge (2002, p.472) says, "this procedure is relatively straightforward and might provide a good estimate of the average effect."

If you insist on estimating a probit, for example, exactly how you should proceed depends on whether your potentially endogenous explanatory variable is continuous or discrete. Furthermore, you need to check what it is you are estimating - for example, in some probit IV models, the output reported has been scaled by a function of the estimated correlation between the residuals.

To give you a flavour of the probit IV model, suppose there is one potentially endogenous, continuous regressor.<sup>5</sup> Start from the latent variable model, written as

$$\begin{aligned}
 y_1^* &= \mathbf{z}_1 \boldsymbol{\delta}_1 + \alpha_1 y_2 + u_1 & (9.3) \\
 y_2 &= \mathbf{z}_1 \boldsymbol{\delta}_{21} + \mathbf{z}_2 \boldsymbol{\delta}_{22} + v_2 = \mathbf{z} \boldsymbol{\delta}_2 + v_2 \\
 y_1 &= 1 [y_1^* > 0].
 \end{aligned}$$

We treat the variable  $y_2$  as potentially endogenous. We assume that  $u_1, v_2$  are normally distributed with constant variances and means equal to zero. Thus we abstract from heteroskedasticity here. Since  $y_1^*$  is unobserved, we cannot identify  $\boldsymbol{\delta}_1$  and  $\text{var}(u_1)$  separately; we therefore normalize by setting the variance of  $u_1$  equal to 1. Note that one implication of  $v_2$  being assumed normally distributed is that  $y_2$  is normally distributed, conditional on  $\mathbf{z}$ . So, within the current framework,  $y_2$  **must** be a continuous variable.

The variables in the vectors  $\mathbf{z}_1$  and  $\mathbf{z}_2$  are assumed exogenous, i.e. they are uncorrelated with the residuals  $u_1, v_2$ . Note that the vector  $\mathbf{z}_2$  are the "instruments" (exclusion restrictions). This implies that  $y_2$  is endogenous in (9.3) - i.e. correlated with  $u_1$  - if and only if the correlation between  $u_1$  and  $v_2$  is

---

<sup>5</sup>For the case where the probit model contains a **binary** endogenous explanatory variable, see Section 15.7.3 in Wooldridge (2002).

non-zero. In other words, if  $u_1, v_2$  follow a bivariate normal distribution,  $y_2$  will be endogenous if and only if the covariance parameter is different from zero.

With this set-up of the model, one way of estimating the unknown parameters is by means of a **two-stage procedure** proposed by Rivers and Vuong (1988). As you will see, this procedure is very similar to the regression-based Hausman test for linear models. Bivariate normality of  $(u_1, v_2)$  with  $Var(u_1) = 1$  implies

$$\begin{aligned} u_1 &= \frac{Cov(v_2, u_1)}{Var(v_2)} v_2 + e_1 \\ u_1 &= \theta_1 v_2 + e_1, \end{aligned}$$

where  $e_1$  is independent of  $\mathbf{z}$  and  $v_2$  (and hence of  $y_2$ ). Notice that  $Var(u_1) = 1$  implies

$$\begin{aligned} Var(e_1) &= 1 - \frac{[Cov(v_2, u_1)]^2}{Var(v_2)} \\ Var(e_1) &= 1 - \frac{Cov(v_2, u_1)}{\sqrt{Var(v_2)}\sqrt{1}} \frac{Cov(v_2, u_1)}{\sqrt{Var(v_2)}\sqrt{1}} \\ Var(e_1) &= 1 - \rho_1^2, \end{aligned}$$

where  $\rho_1 = corr(u_1, v_2)$ . Now re-write the latent variable equation as

$$y_1^* = \mathbf{z}_1 \boldsymbol{\delta}_1 + \alpha_1 y_2 + \theta_1 v_2 + e_1,$$

and notice  $Var(e_1) < 1$ . Now think of  $v_2$  as a variable that we can condition on in the probit regression.

Then clearly

$$\Pr(y_1 = 1 | \mathbf{z}, y_2, v_2) = \Phi \left( \frac{\mathbf{z}_1 \boldsymbol{\delta}_1 + \alpha_1 y_2 + \theta_1 v_2}{\sqrt{1 - \rho_1^2}} \right).$$

That is, probit on  $\mathbf{z}_1, y_2, v_2$  consistently estimates  $\boldsymbol{\delta}_1 / \sqrt{1 - \rho_1^2}$ ,  $\alpha_1 / \sqrt{1 - \rho_1^2}$  and  $\theta_1 / \sqrt{1 - \rho_1^2}$ . Since  $\rho_1^2 < 1$ , each of these estimated coefficients is greater than its unscaled counterpart unless  $y_2$  is exogenous (in which case  $\rho_1^2 = 0$ ). We return to this issue below.

Of course, in practice we do not observe  $v_2$ , but we can estimate it, as follows.

1. Run the OLS regression  $y_2$  on  $\mathbf{z}$  and save the estimated residuals  $\hat{v}_2$
2. Run the probit  $y_1$  on  $\mathbf{z}_1, y_2, \hat{v}_2$ . This consistently estimates the coefficients  $\boldsymbol{\delta}_1/\sqrt{1-\rho_1^2}$ ,  $\alpha_1/\sqrt{1-\rho_1^2}$  and  $\theta_1/\sqrt{1-\rho_1^2}$ .

Notice that under the null hypothesis that  $y_2$  is exogenous, the coefficient on  $\hat{v}_2$  is zero.

As an alternative to the two-stage procedure just outlined, you can estimate

$$\begin{aligned} y_1^* &= \mathbf{z}_1 \boldsymbol{\delta}_1 + \alpha_1 y_2 + u_1 \\ y_2 &= \mathbf{z}_1 \boldsymbol{\delta}_{21} + \mathbf{z}_2 \boldsymbol{\delta}_{22} + v_2 = \mathbf{z} \boldsymbol{\delta}_2 + v_2 \\ y_1 &= 1 [y_1^* > 0]. \end{aligned}$$

simultaneously. This involves writing down the likelihood of observing  $y_1$  and  $y_2$ , and maximizing the implied sample likelihood function (shown in eq. 15.50 in Wooldridge). Unlike the two-step procedure, this approach provides direct estimates of the non-scaled parameters  $\boldsymbol{\delta}_1, \alpha_1$ , and so computing partial effects is straightforward - for example:

$$\frac{\partial \Pr(y_1 = 1 | \mathbf{z}, y_2)}{\partial y_2} = \phi(\mathbf{z}_1 \hat{\boldsymbol{\delta}}_1 + \hat{\alpha}_1 y_2) \hat{\alpha}_1.$$

Another advantage of the simultaneous approach is that it is straightforward to get standard errors on the marginal effects. This is not the case for the two step estimator.

EXAMPLE: Female labour supply and endogenous education. Appendix.

**PhD Programme: Applied Econometrics**  
**Department of Economics, University of Gothenburg**  
**Appendix Lecture 10**  
Måns Söderbom

**Binary Choice Models**

**Application: School Choice in India**

The data used below were kindly provided by Dr Geeta Kingdon. These data have been used in

Kingdon, G. (1996) 'The quality and efficiency of private and public education: a case-study of urban India,' *Oxford Bulletin of Economics and Statistics* 58: 57-81.

See Table 1 in the paper for details on how variables are defined.

Key variables and summary statistics:

Contains data from kingdon96.dta

```
obs:      902
vars:      9
size:     36,080 (99.7% of memory free)
```

variable name	storage type	display format	value label	variable label
numsib	float	%9.0g		Number of siblings
sraven	float	%9.0g		Raven ability score
wealth	float	%9.0g		Index of household asset value
male	float	%9.0g		Gender dummy: male=1, female=0
lowcaste	float	%9.0g		Low caste? yes=1,no=0
muslim	float	%9.0g		Muslim? yes=1,no=0
medyrs	float	%9.0g		Mother's education in years
sikhchr	float	%9.0g		Sikh or Christian? yes=1,no=0
stype	float	%9.0g		School type: 0=govt, 1=private aided, 2=private unaided

Variable	Obs	Mean	Std. Dev.	Min	Max
puaind	902	.3991131	.4899878	0	1
numsib	902	3.988914	1.705215	1	11
sraven	902	30.52661	11.22551	3	57
wealth	902	24.25723	21.08854	.072	82
male	902	.5321508	.4992421	0	1
lowcaste	902	.1330377	.3398039	0	1
muslim	902	.2184035	.4133919	0	1
medyrs	902	8.665188	4.954049	0	20
medyrsq	902	99.60089	79.36289	0	400
sikhchr	902	.0310421	.1735278	0	1

Now consider results from OLS, probit and logit using the Stata code in Box 1.

### Box 1: Stata code for estimation of binary choice models

```
#delimit;

use "F:\teaching_gbg08\Applied_econ08\kingdon96.dta", clear;

describe;
summarize;

tabstat numsib sraven wealth male ,
by(stype) s(mean p50 sd);

tabstat lowcaste muslim medyrs sikhchr,
by(stype) s(mean p50 sd);

ge puaind=stype==2;
replace puaind=. if stype==.;

ge medyrsq = medyrs^2;

sum puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq
sikhchr;

/** LPM, probit, logit */

reg puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq
sikhchr;
reg puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq
sikhchr, robust;
predict yhat; /* obtain predicted probability */
predict xb, xb;
label var xb "xb (index)";
scatter puaind yhat xb, symbol(+ o) jitter(2) lltitle("Linear prediction &
actual outcome");

probit puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq
sikhchr;
predict phat, p; /* obtain predicted probability */

ge phat_d=phat>.5; /* predicted outcome */
tab phat_d puaind; /* compare predicted & actual outcomes */

test sraven wealth; /* Wald test, joint significance */

mfx compute;

logit puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq
sikhchr;
predict lhat, p; /* obtain predicted probability */

sum yhat phat lhat;
count if yhat<0; /* number of negative predicted probabilities, LPM
*/
count if yhat>1; /* number of predicted probabilities in excess of
one, LPM */

exit;
```

**Table 1a. LINEAR PROBABILITY MODEL**

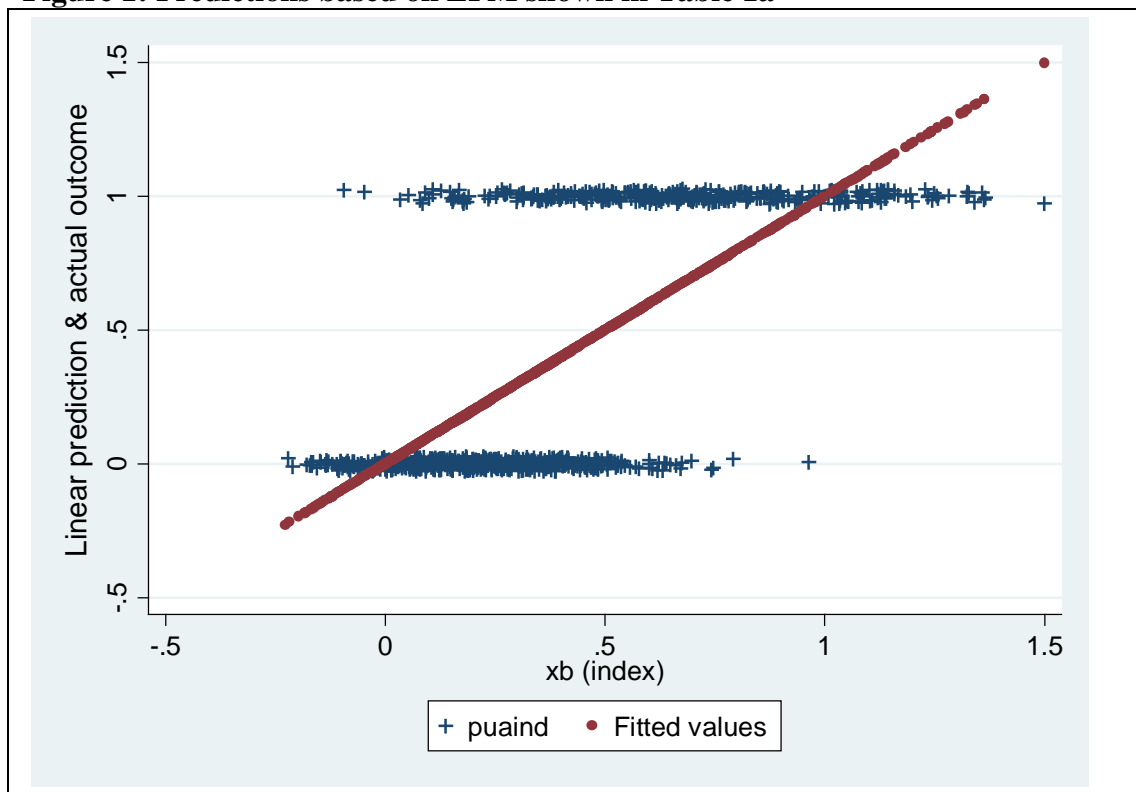
```
> regress puaind numsis sraven wealth male lowcaste muslim medyrs medyrsq sikhchr;
```

Source	SS	df	MS	Number of obs = 902		
Model	100.026088	9	11.1140097	F( 9, 892)	=	85.25
Residual	116.293203	892	.130373546	Prob > F	=	0.0000
				R-squared	=	0.4624
				Adj R-squared	=	0.4570
				Root MSE	=	.36107

puaind	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
numsis	-.0223168	.0082608	-2.70	0.007	-.0385297	-.0061038
sraven	.0075825	.0012103	6.27	0.000	.0052072	.0099578
wealth	.0101314	.0007259	13.96	0.000	.0087067	.0115561
male	.1732116	.0245567	7.05	0.000	.1250159	.2214072
lowcaste	-.1412188	.0392124	-3.60	0.000	-.2181782	-.0642594
muslim	-.1387535	.0321586	-4.31	0.000	-.2018689	-.0756381
medyrs	-.0245589	.0078772	-3.12	0.002	-.0400188	-.0090989
medyrsq	.0016972	.0005077	3.34	0.001	.0007008	.0026937
sikhchr	.220197	.0702409	3.13	0.002	.0823403	.3580538
_cons	.0047471	.0624763	0.08	0.939	-.1178706	.1273648

**Figure 1: Predictions based on LPM shown in Table 1a**



Note: The linear prediction is denoted  $\hat{y}$ , and puaind is the actual binary dependent variable. The puaind variable has been “jittered” to facilitate interpretation.



```

. sum yhat;

Variable |      Obs      Mean   Std. Dev.   Min       Max
-----|-----
yhat    |     902   .3991131   .3331918  -.2283615  1.499489

. count if yhat>1;
61

. count if yhat<0;
81

```

**Table 1b. LINEAR PROBABILITY MODEL WITH STANDARD ERRORS ROBUST TO HETEROSKEDASTICITY**

```

> regress puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq sikhchr,
> robust ;

```

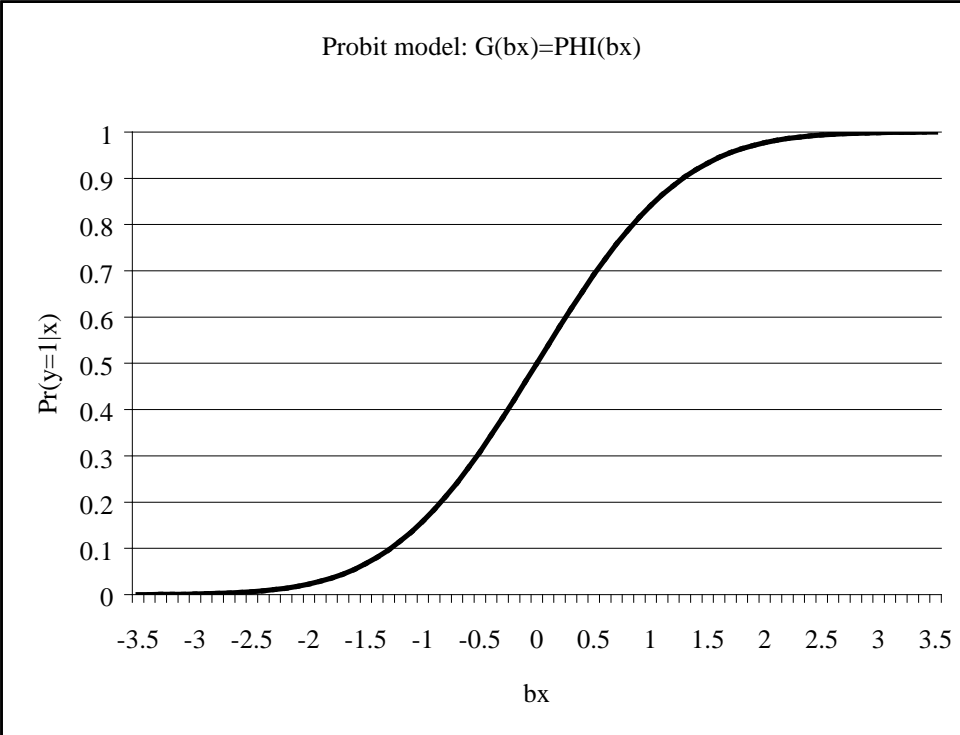
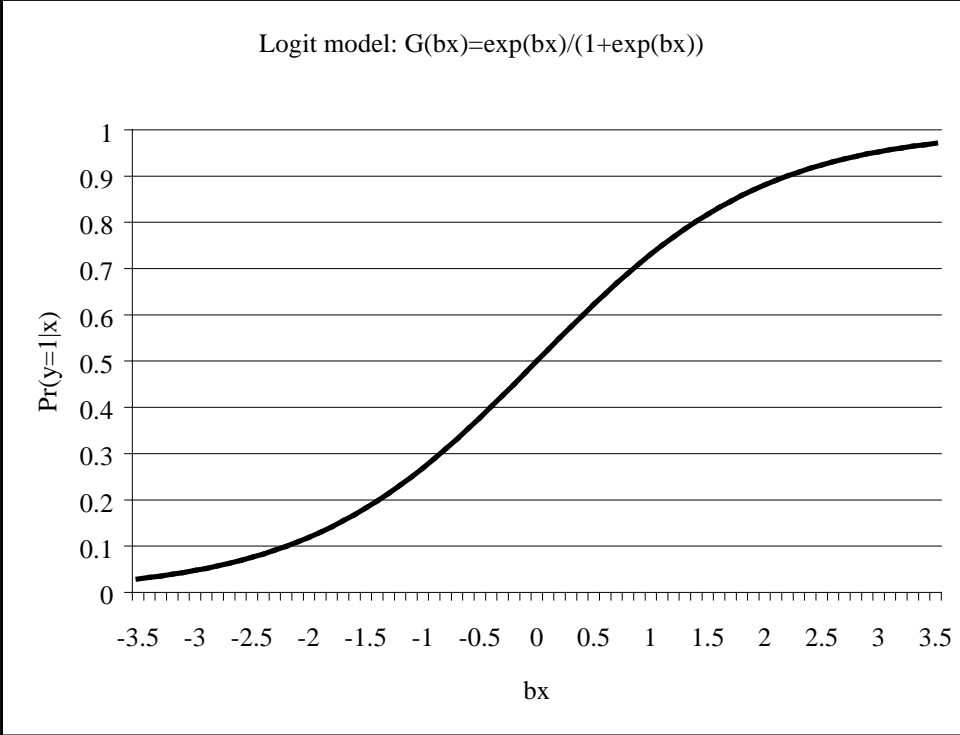
```

Regression with robust standard errors                                Number of obs =      902
                                                                    F( 9, 892) = 189.05
                                                                    Prob > F      = 0.0000
                                                                    R-squared     = 0.4624
                                                                    Root MSE     = .36107

```

puaind	Coef.	Robust Std. Err.	t	P> t	[95% Conf. Interval]	
numsib	-.0223168	.0084851	-2.63	0.009	-.0389699	-.0056636
sraven	.0075825	.00126	6.02	0.000	.0051096	.0100554
wealth	.0101314	.0006368	15.91	0.000	.0088817	.0113811
male	.1732116	.0241983	7.16	0.000	.1257193	.2207039
lowcaste	-.1412188	.0374195	-3.77	0.000	-.2146593	-.0677782
muslim	-.1387535	.0317488	-4.37	0.000	-.2010645	-.0764425
medyrs	-.0245589	.007757	-3.17	0.002	-.039783	-.0093347
medyrsq	.0016972	.000503	3.37	0.001	.00071	.0026845
sikhchr	.220197	.0775071	2.84	0.005	.0680796	.3723145
_cons	.0047471	.0618058	0.08	0.939	-.1165547	.1260488

**Figure 2: The Logit and Probit CDFs**



**Table 2. PROBIT MODEL**

```
> probit puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq sikhchr;
```

```
Iteration 0: log likelihood = -606.73067
Iteration 1: log likelihood = -373.10677
Iteration 2: log likelihood = -343.27331
Iteration 3: log likelihood = -340.47774
Iteration 4: log likelihood = -340.43889
Iteration 5: log likelihood = -340.43888
```

```
Probit estimates                               Number of obs   =           902
                                                LR chi2(9)      =           532.58
                                                Prob > chi2     =           0.0000
Log likelihood = -340.43888                    Pseudo R2      =           0.4389
```

puaind	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
numsib	-.0998298	.0382835	-2.61	0.009	-.1748641 -.0247956
sraven	.0301986	.0054653	5.53	0.000	.0194869 .0409103
wealth	.0461453	.0043108	10.70	0.000	.0376963 .0545943
male	.8575159	.1199153	7.15	0.000	.6224862 1.092546
lowcaste	-.5496526	.1865875	-2.95	0.003	-.9153575 -.1839478
muslim	-.7229197	.1530685	-4.72	0.000	-1.022929 -.4229109
medyrs	-.1260082	.0373075	-3.38	0.001	-.1991296 -.0528868
medyrsq	.0079365	.0024278	3.27	0.001	.0031781 .0126948
sikhchr	.8875504	.3272338	2.71	0.007	.246184 1.528917
_cons	-1.882662	.287822	-6.54	0.000	-2.446783 -1.318541

```
. /* marginal effects using mfx compute */
> mfx compute;
```

```
Marginal effects after probit
y = Pr(puaind) (predict)
= .38659838
```

variable	dy/dx	Std. Err.	z	P> z	[ 95% C.I. ]	X
numsib	-.0382063	.01465	-2.61	0.009	-.066911 -.009502	3.98891
sraven	.0115574	.00208	5.54	0.000	.007472 .015643	30.5266
wealth	.0176605	.00172	10.27	0.000	.014289 .021032	24.2572
male*	.3167018	.04152	7.63	0.000	.235328 .398076	.532151
lowcaste*	-.1926379	.05762	-3.34	0.001	-.305563 -.079713	.133038
muslim*	-.2513949	.04612	-5.45	0.000	-.341793 -.160997	.218404
medyrs	-.0482251	.01433	-3.37	0.001	-.076308 -.020142	8.66519
medyrsq	.0030374	.00093	3.26	0.001	.001211 .004864	99.6009
sikhchr*	.3401752	.11123	3.06	0.002	.122159 .558191	.031042

(\*) dy/dx is for discrete change of dummy variable from 0 to 1

**Table 3. LOGIT MODEL**

```
> logit puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq sikhchr;
```

```
Iteration 0: log likelihood = -606.73067
Iteration 1: log likelihood = -373.59867
Iteration 2: log likelihood = -342.99987
Iteration 3: log likelihood = -338.7387
Iteration 4: log likelihood = -338.60417
Iteration 5: log likelihood = -338.604
```

```
Logit estimates                               Number of obs   =       902
                                                LR chi2(9)     =       536.25
                                                Prob > chi2    =       0.0000
Log likelihood = -338.604                    Pseudo R2      =       0.4419
```

puaind	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
numsib	-.1763513	.0702654	-2.51	0.012	-.3140689 -.0386336
sraven	.0566268	.0099437	5.69	0.000	.0371376 .0761161
wealth	.0812388	.0079356	10.24	0.000	.0656852 .0967923
male	1.573975	.2206061	7.13	0.000	1.141595 2.006355
lowcaste	-1.097276	.35503	-3.09	0.002	-1.793122 -.4014304
muslim	-1.30201	.2766134	-4.71	0.000	-1.844162 -.7598576
medyrs	-.2326064	.0666913	-3.49	0.000	-.363319 -.1018938
medyrsq	.0142063	.0043246	3.28	0.001	.0057302 .0226824
sikhchr	1.672074	.5791094	2.89	0.004	.5370403 2.807107
_cons	-3.388287	.5269294	-6.43	0.000	-4.421049 -2.355524

```
. /* marginal effects using mfx compute */
> mfx compute;
```

```
Marginal effects after logit
y = Pr(puaind) (predict)
= .36959733
```

variable	dy/dx	Std. Err.	z	P> z	[ 95% C.I. ]	X
numsib	-.041089	.01632	-2.52	0.012	-.073077 -.009101	3.98891
sraven	.0131938	.00229	5.76	0.000	.008701 .017686	30.5266
wealth	.0189282	.00196	9.65	0.000	.015082 .022774	24.2572
male*	.3480658	.04426	7.86	0.000	.261309 .434823	.532151
lowcaste*	-.2195702	.05702	-3.85	0.000	-.331332 -.107809	.133038
muslim*	-.26307	.04577	-5.75	0.000	-.352781 -.173359	.218404
medyrs	-.0541962	.01563	-3.47	0.001	-.084826 -.023567	8.66519
medyrsq	.00331	.00101	3.27	0.001	.001324 .005296	99.6009
sikhchr*	.3900823	.10997	3.55	0.000	.174542 .605622	.031042

(\*) dy/dx is for discrete change of dummy variable from 0 to 1

**Table 4. PREDICTED PROBABILITIES: PROBIT AND LOGIT**

```
. sum phat lhat;
```

Variable	Obs	Mean	Std. Dev.	Min	Max
phat	902	.4023303	.3436492	.0009482	.999997
lhat	902	.3991131	.3473132	.0037879	.9996895

Note: phat = predicted probability based on probit model; lhat = predicted probability based on logit model.

**Table 5. SIMPLE PROBIT MODEL: PUA = PHI(SRAVEN)**

```

Probit estimates                               Number of obs   =       902
                                                LR chi2(1)      =      152.66
                                                Prob > chi2     =       0.0000
Log likelihood = -530.40283                    Pseudo R2      =       0.1258
    
```

puaind	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
sraven	.0499438	.0042474	11.76	0.000	.0416191	.0582685
_cons	-1.816083	.1414252	-12.84	0.000	-2.093271	-1.538894

**Table 6: The baseline probit model (same as Table 2)**

```

Probit estimates                               Number of obs   =       902
                                                LR chi2(9)      =      532.58
                                                Prob > chi2     =       0.0000
Log likelihood = -340.43888                    Pseudo R2      =       0.4389
    
```

puaind	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
numsib	-.0998298	.0382835	-2.61	0.009	-.1748641	-.0247956
sraven	.0301986	.0054653	5.53	0.000	.0194869	.0409103
wealth	.0461453	.0043108	10.70	0.000	.0376963	.0545943
male	.8575159	.1199153	7.15	0.000	.6224862	1.092546
lowcaste	-.5496526	.1865875	-2.95	0.003	-.9153575	-.1839478
muslim	-.7229197	.1530685	-4.72	0.000	-1.022929	-.4229109
medyrs	-.1260082	.0373075	-3.38	0.001	-.1991296	-.0528868
medyrsq	.0079365	.0024278	3.27	0.001	.0031781	.0126948
sikhchr	.8875504	.3272338	2.71	0.007	.246184	1.528917
_cons	-1.882662	.287822	-6.54	0.000	-2.446783	-1.318541

```
. test sraven wealth;
```

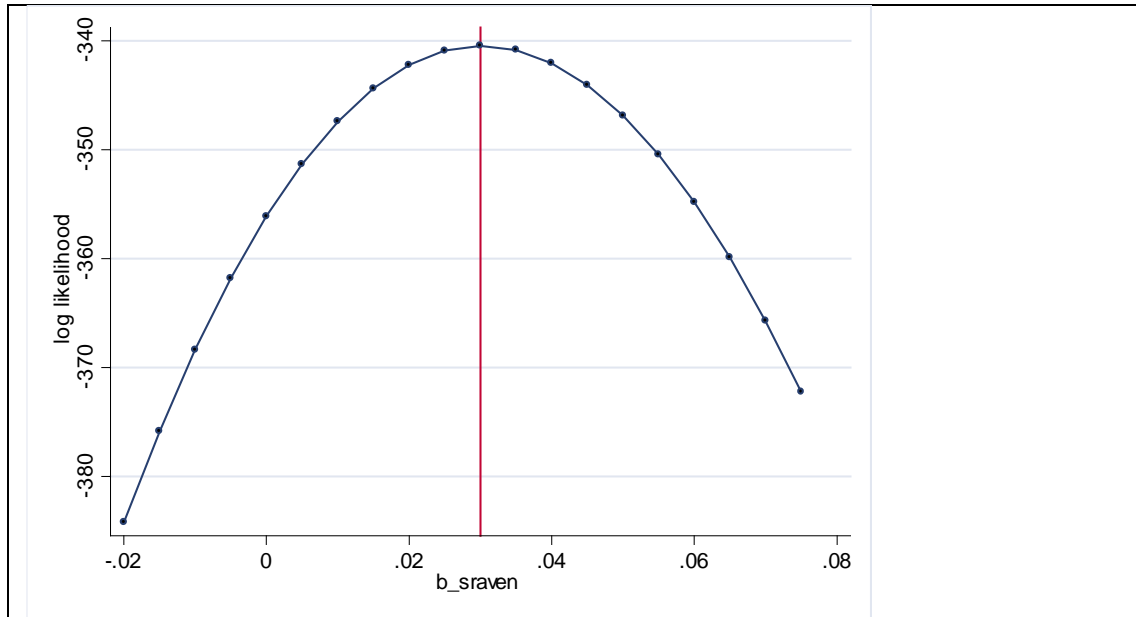
```
( 1) sraven = 0
( 2) wealth = 0
```

```

      chi2( 2) = 150.74
      Prob > chi2 = 0.0000
    
```

Now vary the coefficient on *sraven* around the ML estimate of 0.03 - see Figure 1.

Figure 3: The log-likelihood as function of b\_sraven



As expected, values of b\_sraven not equal to 0.03 produce a lower log likelihood value. Is it important how much the log L falls as a result of moving b\_sraven away from the ML estimate of 0.03?

Predictions:

```
. predict phat, p;
. ge phat_d=phat>.5;
. table phat_d pua;
```

**Table 7: Frequencies of correct predictions**

phat_d	puaind	
	0	1
0	491	87
1	51	273

*Illustration of LR test*

Estimate restricted model without *sraven* and *wealth*. Compare the resulting log likelihood value to that obtained in the unrestricted model (Table 2):

**Table 8: Restricted probit: *sraven* and *wealth* omitted**

Probit estimates	Number of obs	=	902
	LR chi2(7)	=	311.29
	Prob > chi2	=	0.0000
Log likelihood = -451.08324	Pseudo R2	=	0.2565

```
-----+-----
```

puaind	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
numsib	-.1092096	.0343527	-3.18	0.001	-.1765395 -.0418796
male	.6587308	.1010961	6.52	0.000	.4605861 .8568756
lowcaste	-.5847738	.1689565	-3.46	0.001	-.9159225 -.253625
muslim	-.6309888	.1307152	-4.83	0.000	-.8871859 -.3747918
medyrs	-.0969399	.0336861	-2.88	0.004	-.1629634 -.0309163
medyrsq	.0127564	.0021721	5.87	0.000	.0084992 .0170135
sikhchr	.8240591	.3017656	2.73	0.006	.2326093 1.415509
_cons	-.4723225	.2159322	-2.19	0.029	-.8955419 -.0491031

```
-----+-----
```

```
. display 2*(-340.43888 - -451.08324 )
221.28872
```

```
. disp chiprob(2,221.29)
8.861e-49
```

```
(=0.0000000000...)
```

## Box 2: Stata code for functional form tests: Probit

```
use kingdon96.dta, clear;

ge puaind=sttype==2;
ge medyrsq = medyrs^2;

probit puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq
sikhchr;

predict xb, xb;
predict pprobit, p;

/* first do parametric test */

ge xb2=xb^2;
ge xb3=xb^3;

probit puaind xb2 xb3, offset(xb) nocons;

test xb2 xb3;

/* wald test of the hypothesis that the coefficients on xb2 and xb3 are
equal to zero */

/* next do semiparametric test */

lowess puaind xb, gen(psempar) bw(0.4);

/* use quite a small bandwidth or the estimated function will
'oversmooth' the data. Stata's default is 0.8 */
label var xb "xb";
label var psempar "f(xb)";
label var pprobit "Probit Pr(puaind=1|xb)";

scatter puaind psempar pprobit xb, symbol(+ o i) connect(. . 1)
sort(pprobit) jitter(2);

exit;
```



Figure 4: Illustration of *lowess* smoothing

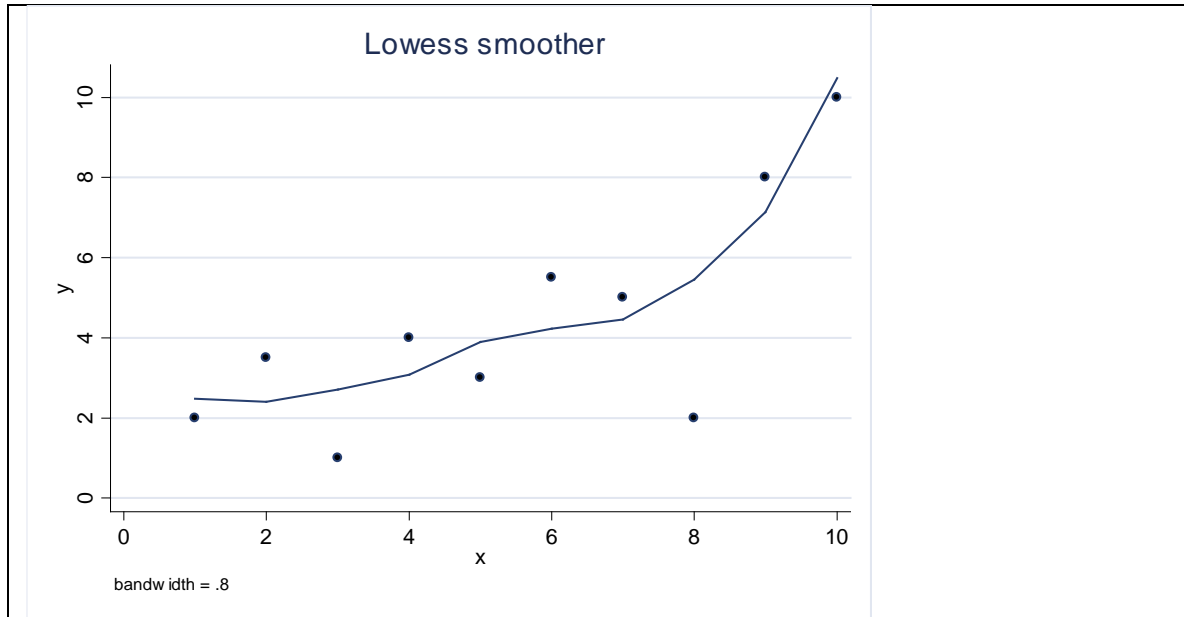
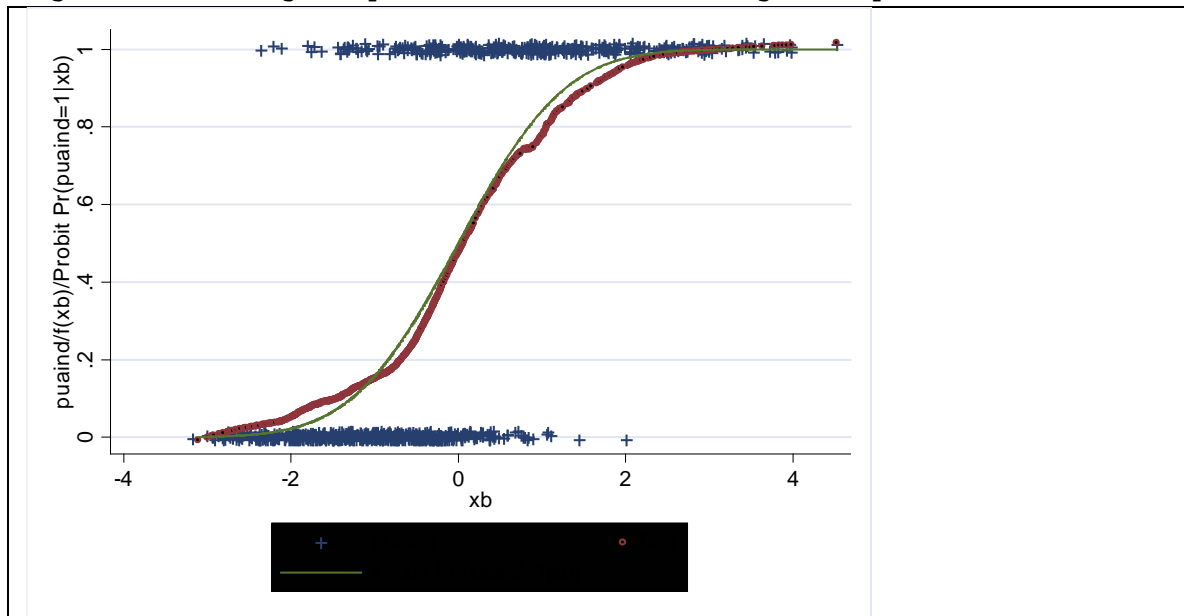


Figure 5: Assessing the probit functional form using a semiparametric method



Note:  $f(xb)$  is a semiparametric estimate obtained by means of regressing *pua ind* on *xb* using the *lowess* command with bandwidth = 0.4. Actual outcomes 0 or 1 (*pua ind*) are plotted after jittering for readability. See Box 2 for details on how this was coded in Stata.

```

ge xb2=xb^2;
. ge xb3=xb^3;
. probit puaind xb2 xb3, offset(xb) nocons;

```

**Table 9: Alternative model, containing a nonlinear function of xb**

```

Probit estimates                               Number of obs =          902
                                                Wald chi2(2) =           3.75
Log likelihood = -338.70049                    Prob > chi2 =           0.1536

```

puaind	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
xb2	.0622552	.0453367	1.37	0.170	-.0266031	.1511134
xb3	-.0017062	.0226679	-0.08	0.940	-.0461344	.042722
xb	(offset)					

Notice the two options used here (the instructions after the , in probit):  
offset(xb) is equivalent to imposing a coefficient equal to one on xb, and nocons  
means the model is estimated without a constant.

Wald test:

```

. test xb2 xb3;

( 1)  xb2 = 0
( 2)  xb3 = 0

           chi2( 2) =    3.75
           Prob > chi2 =    0.1536

```

Log likelihood ratio test:

```

disp 2*( -338.70049    --340.43888    )
3.47678

disp chiprob(2,3.47678)
.17580322

```

## Heteroskedasticity in the school choice probit

Consider the benchmark probit model reported in Table 2 above:

```

Probit estimates                                     Number of obs   =      902
                                                    LR chi2(9)      =     532.58
                                                    Prob > chi2     =      0.0000
Log likelihood = -340.43888                         Pseudo R2       =      0.4389
  
```

	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
puaind					
numsib	-.0998298	.0382835	-2.61	0.009	-.1748641  -.0247956
sraven	.0301986	.0054653	5.53	0.000	.0194869  .0409103
wealth	.0461453	.0043108	10.70	0.000	.0376963  .0545943
male	.8575159	.1199153	7.15	0.000	.6224862  1.092546
lowcaste	-.5496526	.1865875	-2.95	0.003	-.9153575  -.1839478
muslim	-.7229197	.1530685	-4.72	0.000	-1.022929  -.4229109
medyrs	-.1260082	.0373075	-3.38	0.001	-.1991296  -.0528868
medyrsq	.0079365	.0024278	3.27	0.001	.0031781  .0126948
sikhchr	.8875504	.3272338	2.71	0.007	.246184  1.528917
_cons	-1.882662	.287822	-6.54	0.000	-2.446783  -1.318541

Now relax the assumption that the error term is homoskedastic, by writing the variance of the error term as  $[\exp(g*sraven)]^2$ , where  $g$  is a parameter to be estimated (note: if  $g=0$  we're back to the homoskedastic model). I can obtain results for this generalized model by using the `hetprob` command in Stata:

```

hetprob puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq
sikhchr, het(sraven);
  
```

```

Heteroskedastic probit model                       Number of obs   =      902
                                                    Zero outcomes   =      542
                                                    Nonzero outcomes =      360
                                                    Wald chi2(9)    =      36.56
Log likelihood = -336.3521                         Prob > chi2     =      0.0000
  
```

	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
puaind					
numsib	-.0672515	.0246799	-2.72	0.006	-.1156232  -.0188798
sraven	.0229242	.0041345	5.54	0.000	.0148208  .0310276
wealth	.0270485	.0057435	4.71	0.000	.0157914  .0383056
male	.5055253	.1190219	4.25	0.000	.2722466  .7388039
lowcaste	-.3304765	.127884	-2.58	0.010	-.5811245  -.0798284
muslim	-.410586	.1214911	-3.38	0.001	-.6487041  -.1724678
medyrs	-.0691932	.0268416	-2.58	0.010	-.1218016  -.0165847
medyrsq	.0041276	.0017081	2.42	0.016	.0007798  .0074754
sikhchr	.4802327	.2136518	2.25	0.025	.0614829  .8989825
_cons	-1.272155	.2532967	-5.02	0.000	-1.768607  -.7757022
lnsigma2					
sraven	-.0171179	.0059309	-2.89	0.004	-.0287422  -.0054935

```

Likelihood-ratio test of lnsigma2=0: chi2(1) =      8.17  Prob > chi2 = 0.0043
  
```

Clearly there is evidence here that the variance of the error term falls with `sraven`. Alternatively, this can be interpreted as indicating that the functional form of the baseline probit is wrong. Now consider adding **`sraven squared`** to the baseline model, on the grounds that this is a generalization of the baseline probit. Results:

```

. ge sraven2=sraven^2;

. probit puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq sikhchr
sraven2;

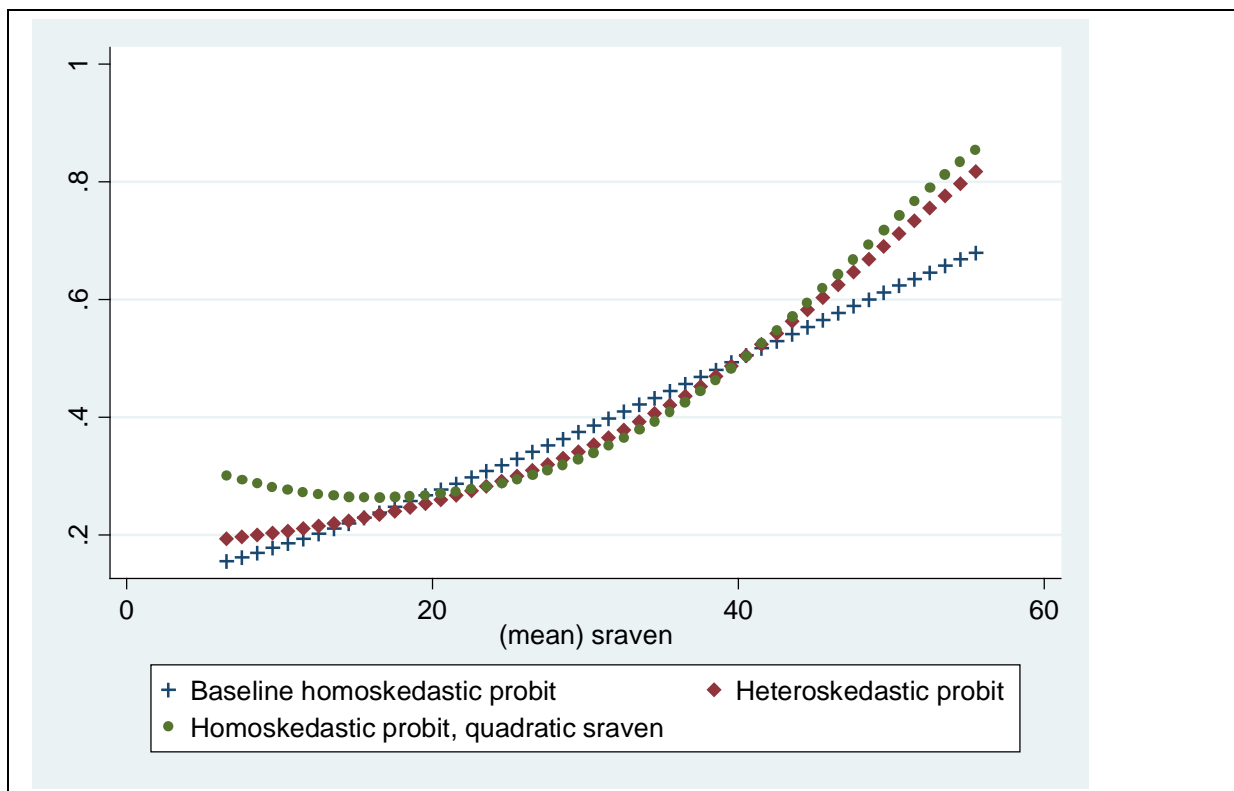
Probit regression                               Number of obs   =       902
                                                LR chi2(10)    =       539.44
                                                Prob > chi2    =       0.0000
Log likelihood = -337.00859                    Pseudo R2      =       0.4445

```

puaind	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
numsib	-.0976174	.0383671	-2.54	0.011	-.1728155 - .0224192
sraven	-.0364237	.0257712	-1.41	0.158	-.0869344 .0140869
wealth	.0467619	.0044055	10.61	0.000	.0381273 .0553964
male	.8598127	.1210329	7.10	0.000	.6225926 1.097033
lowcaste	-.5200105	.1866679	-2.79	0.005	-.8858729 -.1541481
muslim	-.7285526	.1536653	-4.74	0.000	-1.029731 -.4273742
medyrs	-.1232646	.0372906	-3.31	0.001	-.1963527 -.0501765
medyrsq	.00768	.0024345	3.15	0.002	.0029084 .0124516
sikhchr	.9275063	.3282048	2.83	0.005	.2842368 1.570776
sraven2	.0011059	.0004214	2.62	0.009	.00028 .0019317
_cons	-1.0303	.4258538	-2.42	0.016	-1.864958 -.195642

Clearly the squared term is quite significant.

Based on the three models shown above, the following graph illustrates how the predicted probability of going to a private unaided school varies with sraven, holding all other explanatory factors constant.



It seems the heteroskedastic probit and the homoscedastic probit with  $\text{sraven}^2$  included tell a similar story: the likelihood that  $y=1$  is relatively insensitive to changes to  $\text{sraven}$  at low levels, but more sensitive to changes to  $\text{sraven}$  at high levels than what is implied by the benchmark model.

**Box 3: Stata code generating the graph on the previous page**

```
probit puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq
sikhchr;
estimates store base;
hetprob puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq
sikhchr, het(sraven);
estimates store het;

ge sraven2=sraven^2;
probit puaind numsib sraven wealth male lowcaste muslim medyrs medyrsq
sikhchr sraven2;
estimates store quad;

collapse _all;
ge junk=50;
expand junk;
replace sraven=sraven+(_n-25);
replace sraven2=sraven^2;

estimates restore base;
predict p1, p;
estimates restore het;
predict p2, p;
estimates restore quad;
predict p3, p;

label var p1 "Baseline homoskedastic probit";
label var p2 "Heteroskedastic probit";
label var p3 "Homoskedastic probit, quadratic sraven";

scatter p1 p2 p3 sraven, s(+ d o);
exit;
```

## A probit model of female labour supply with endogenous education

This section uses the MROZ dataset.<sup>1</sup> Variable names are obvious except *nwifeinc*, which measures non-wife income in thousands.

```
. use C:\teaching_gbg07\applied_econ07\MROZ.dta
```

```
. Simple probit assuming all regressors exogenous
```

```
. probit inlf nwifeinc educ exper expersq age kidslt6 kidsge6
```

```
Iteration 0: log likelihood = -514.8732
Iteration 1: log likelihood = -405.78215
Iteration 2: log likelihood = -401.32924
Iteration 3: log likelihood = -401.30219
Iteration 4: log likelihood = -401.30219
```

```
Probit regression                               Number of obs   =           753
                                                LR chi2(7)      =           227.14
                                                Prob > chi2     =           0.0000
Log likelihood = -401.30219                    Pseudo R2      =           0.2206
```

inlf	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
nwifeinc	-.0120237	.0048398	-2.48	0.013	-.0215096 -.0025378
educ	.1309047	.0252542	5.18	0.000	.0814074 .180402
exper	.1233476	.0187164	6.59	0.000	.0866641 .1600311
expersq	-.0018871	.0006	-3.15	0.002	-.003063 -.0007111
age	-.0528527	.0084772	-6.23	0.000	-.0694678 -.0362376
kidslt6	-.8683285	.1185223	-7.33	0.000	-1.100628 -.636029
kidsge6	.036005	.0434768	0.83	0.408	-.049208 .1212179
_cons	.2700768	.508593	0.53	0.595	-.7267472 1.266901

```
. mfx compute, predict(p) nose
```

```
Marginal effects after probit
y = Pr(inlf) (predict, p)
= .58154201
```

variable	dy/dx	X
nwifeinc	-.0046962	20.129
educ	.0511287	12.2869
exper	.0481771	10.6308
expersq	-.0007371	178.039
age	-.0206432	42.5378
kidslt6	-.3391514	.237716
kidsge6	.0140628	1.35325

<sup>1</sup> See examples 15.1 and 15.3 in Wooldridge (2002). Original source of data: Mroz, T.A. (1987) "The sensitivity of an empirical model of married women's hours of work to economic and statistical assumptions," *Econometrica* 55, 765-799.

2-step procedure to allow for endogenous education

```
. reg educ nwifeinc exper expersq age kidslt6 kidsge6 motheduc fatheduc huseduc
```

Source	SS	df	MS	Number of obs =	753
Model	1849.07781	9	205.45309	F( 9, 743) =	74.07
Residual	2060.96203	743	2.77383853	Prob > F =	0.0000
Total	3910.03984	752	5.19952106	R-squared =	0.4729
				Adj R-squared =	0.4665
				Root MSE =	1.6655

educ	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
nwifeinc	.0156893	.0058267	2.69	0.007	.0042506 .027128
exper	.0577544	.0220604	2.62	0.009	.0144462 .1010625
expersq	-.000784	.000721	-1.09	0.277	-.0021994 .0006314
age	-.0059011	.0098709	-0.60	0.550	-.0252792 .013477
kidslt6	.1195954	.1307071	0.91	0.360	-.1370038 .3761945
kidsge6	-.0731404	.0515299	-1.42	0.156	-.174302 .0280212
motheduc	.1300347	.0225669	5.76	0.000	.0857322 .1743373
fatheduc	.0950702	.0214618	4.43	0.000	.0529373 .1372032
huseduc	.3475092	.0235063	14.78	0.000	.3013626 .3936558
_cons	5.43695	.5873755	9.26	0.000	4.283837 6.590064

```
. predict v2, res
```

```
. probit inlf educ nwifeinc exper expersq age kidslt6 kidsge6 v2
```

```
Iteration 0: log likelihood = -514.8732
Iteration 1: log likelihood = -405.39051
Iteration 2: log likelihood = -400.95247
Iteration 3: log likelihood = -400.92551
Iteration 4: log likelihood = -400.92551
```

```
Probit regression                                Number of obs = 753
LR chi2(8) = 227.90
Prob > chi2 = 0.0000
Pseudo R2 = 0.2213
Log likelihood = -400.92551
```

inlf	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
educ	.1035752	.0403061	2.57	0.010	.0245767 .1825737
nwifeinc	-.0102851	.0052347	-1.96	0.049	-.020545 -.0000253
exper	.1262477	.0190256	6.64	0.000	.0889582 .1635373
expersq	-.0019432	.0006032	-3.22	0.001	-.0031254 -.0007609
age	-.0543808	.0086633	-6.28	0.000	-.0713605 -.0374012
kidslt6	-.8630859	.1187394	-7.27	0.000	-1.095811 -.630361
kidsge6	.0313802	.0437901	0.72	0.474	-.0544468 .1172071
v2	.0433658	.050021	0.87	0.386	-.0546736 .1414051
_cons	.6209105	.6497413	0.96	0.339	-.652559 1.89438

Conclusion: I do not reject H0: exogeneity at conventional levels of significance.

Simultaneous estimation of the ivprobit model

```
. ivprobit inlf nwifeinc exper expersq age kidslt6 kidsge6 (educ= /*nwifeinc exper
expersq age kidslt6 kidsge6*/
> motheduc fatheduc huseduc), first
```

Fitting exogenous probit model

```
Iteration 0: log likelihood = -514.8732
Iteration 1: log likelihood = -405.78215
Iteration 2: log likelihood = -401.32924
Iteration 3: log likelihood = -401.30219
Iteration 4: log likelihood = -401.30219
```

Fitting full model

```
Iteration 0: log likelihood = -1848.8468
Iteration 1: log likelihood = -1848.4692
Iteration 2: log likelihood = -1848.468
Iteration 3: log likelihood = -1848.468
```

```
Probit model with endogenous regressors          Number of obs   =          753
Wald chi2(7)                                     =          166.92
Log likelihood = -1848.468                       Prob > chi2      =          0.0000
```

	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
-----						
inlf						
educ	.1031572	.0407985	2.53	0.011	.0231936	.1831208
nwifeinc	-.0102491	.0052533	-1.95	0.051	-.0205455	.0000473
exper	.1259361	.0188817	6.67	0.000	.0889286	.1629435
expersq	-.0019384	.0006013	-3.22	0.001	-.003117	-.0007599
age	-.0542481	.0085909	-6.31	0.000	-.071086	-.0374102
kidslt6	-.8608266	.1189176	-7.24	0.000	-1.093901	-.6277524
kidsge6	.0312766	.043758	0.71	0.475	-.0544876	.1170408
_cons	.6212242	.6472555	0.96	0.337	-.6473733	1.889822
-----						
educ						
nwifeinc	.01564	.0057872	2.70	0.007	.0042973	.0269827
exper	.0576687	.0219132	2.63	0.008	.0147196	.1006178
expersq	-.0007821	.0007162	-1.09	0.275	-.0021858	.0006215
age	-.0059553	.0098058	-0.61	0.544	-.0251743	.0132637
kidslt6	.1192229	.1298363	0.92	0.358	-.1352517	.3736974
kidsge6	-.0732584	.0511872	-1.43	0.152	-.1735834	.0270666
motheduc	.1290786	.0224259	5.76	0.000	.0851246	.1730326
fatheduc	.0948204	.0212956	4.45	0.000	.0530818	.1365591
huseduc	.3485603	.0233304	14.94	0.000	.3028335	.3942871
_cons	5.438992	.5835185	9.32	0.000	4.295317	6.582668
-----						
/athrho	.0720741	.0828432	0.87	0.384	-.0902956	.2344439
/lnsigma	.5034343	.0257685	19.54	0.000	.4529288	.5539397
-----						
rho	.0719496	.0824144			-.090051	.2302409
sigma	1.654393	.0426313			1.572912	1.740095
-----						

```
Instrumented: educ
Instruments: nwifeinc exper expersq age kidslt6 kidsge6 motheduc fatheduc
huseduc
```

```
Wald test of exogeneity (/athrho = 0): chi2(1) = 0.76 Prob > chi2 = 0.3843
```



. mfx compute, predict(p)

Marginal effects after ivprobit

y = Probability of positive outcome (predict, p)  
= .58167552

variable	dy/dx	Std. Err.	z	P> z	[	95% C.I.	]	X
educ	.0402882	.01593	2.53	0.011	.009066	.07151		12.2869
nwifeinc	-.0040028	.00205	-1.95	0.051	-.008024	.000018		20.129
exper	.0491846	.00739	6.65	0.000	.034697	.063672		10.6308
expersq	-.0007571	.00024	-3.22	0.001	-.001218	-.000296		178.039
age	-.0211867	.00335	-6.32	0.000	-.027757	-.014617		42.5378
kidslt6	-.3361976	.04651	-7.23	0.000	-.427348	-.245047		.237716
kidsge6	.0122151	.01709	0.71	0.475	-.021287	.045717		1.35325
motheduc	0	0	.	.	0	0		9.251
fatheduc	0	0	.	.	0	0		8.80876
huseduc	0	0	.	.	0	0		12.4914

.